Hidden Consequence of Active Local Lorentz Invariance*

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Abstract

In this paper we investigate a hidden consequence of the hypothesis that Lagrangians and field equations must be invariant under active local Lorentz transformations. We show that this hypothesis implies in an equivalence between spacetime structures with several curvature and torsion possibilities.

1 Introduction

It is now a well established fact that Maxwell, Dirac and Einstein theories can be formulated in terms of differential forms. The easiest way to see this is to introduce the Clifford and spin-Clifford bundles [28, 25, 14] of spacetime. This formalism for the case of Maxwell and Dirac fields will be briefly recalled in Section 2, since they will play an essential role for proving the main claim of this paper. For the case of the gravitational field, see [26] and also [23, 29, 30].

As it is well known [29], any field theory formulated in terms of differential forms is such that the action is invariant under arbitrary diffeomorphisms. This does not imply that the field equations of the theory are necessarily invariant under diffeomorphisms, *unless* we are prepared to accept as equivalent different manifolds equipped with metrics and connections which may be said diffeomorphically equivalent [1]. Now, several authors, e.g., [13, 22] insist that the invariance of the action (and field equations) under arbitrary active Lorentz

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[1]A thoughtful discussion on this issue will be presented elsewhere.

transformations is a necessary consequence of the equivalence principle[2]. From the mathematical point of view, in order to have such imposition satisfied it is necessary to introduce the concept of *generalized gauge covariant derivatives* (Section 3 and Appendix D). We show that once we accept this concept we must necessarily also accept the equivalence between spacetime models with *connections* that have *different* curvature and torsion tensors. This paper is organized as follows. In Section 2 we introduce the Clifford and spin-Clifford bundles and discuss the concept of Dirac-Hestenes spinor fields (which are sections of a spin-Clifford bundle) and their *representatives* in the Clifford bundle, which are sums of even nonhomogenous differential forms. In Section 3 we discuss the covariant derivative of Clifford fields and introduce the curvature and torsion extensor fields of a given connection. In Section 4 we discuss the covariant derivative of spinor fields, and how to define an *effective* covariant derivative for representatives of Dirac-Hestenes spinor fields in the Clifford bundle. In Section 5 we discuss the many faces of Dirac equation. In Section 6 we discuss the meaning of active local Lorentz invariance of Maxwell equations. In Section 7, we introduce the Dirac-Hestenes equation in a Riemann-Cartan spacetime, and in Section 8 we discuss the meaning of active local Lorentz invariance of the Dirac-Equation. We prove then our main result, namely: active local Lorentz invariance implies a gauge identification of geometries with connections that have different torsion and curvature tensors. The intelligibility of the issues discussed in this paper requires a working knowledge of the theory of connections, and in particular, the introduction of the concept of generalized G-connections. The main results needed are presented in Appendices A-D. This paper is dedicated to Ivanenko, that in a pioneer paper [10] used sums of nonhomogeneous antisymmetric tensors to represent spinors. The way in which such an idea becomes mathematically legitimate is presented below.

2 Clifford and Spin-Clifford Bundles

Let $\mathcal{M} = (M, \mathtt{g}, \nabla, \tau_{\mathtt{g}}, \uparrow)$ be an arbitrary Riemann-Cartan spacetime. The quadruple $(M, \mathtt{g}, \tau_{\mathtt{g}}, \uparrow)$ is a four-dimensional time-oriented and space-oriented Lorentzian manifold. This means that $\mathtt{g} \in \sec T_2^0 M$ is a Lorentzian metric of signature (1,3), $\tau_{\mathtt{g}} \in \sec \bigwedge^4(T^*M)$ and $\uparrow$ is a time-orientation (see details, e.g., in [27]). Here, T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T_x^*M$, $TM = \cup_{x\in M} T_xM$, and $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space[3]. ∇ is an arbitrary metric compatible connection *i.e.*, $\nabla \mathtt{g} = 0$, but in general, $\mathbf{R}(\nabla) \neq 0$, $\mathbf{T}(\nabla) \neq 0$, $\mathbf{R}$ and $\mathbf{T}$ being respectively the curvature and torsion tensors. When $\mathbf{R}(\nabla) \neq 0$, $\mathbf{T}(\nabla) \neq 0$, $\mathcal{M}$ is called a *Riemann-Cartan spacetime*. When $\mathbf{R}(\nabla) \neq 0$, $\mathbf{T}(\nabla) = 0$, $\mathcal{M}$ is called a *Lorentzian spacetime*. When $\mathbf{R}(\nabla) = 0$, $\mathbf{T}(\nabla) \neq 0$, $\mathcal{M}$ is called a *teleparallel spacetime*. Minkowski spacetime is the case where $\mathbf{R}(\nabla) = 0$, $\mathbf{T}(\nabla) = 0$, and $M \simeq \mathbb{R}^4$. Let

[2] We are not going to enter discussions about the equivalence principle in this paper. One of the authors' view on the subject is discussed in [24].

[3] Not to be confused with Minkowski spacetime [27].

$g \in \sec T_0^2 M$ be the metric of the *cotangent bundle*. The Clifford bundle of differential forms $\mathcal{C}\ell(M,g)$ is the bundle of algebras, i.e., $\mathcal{C}\ell(M,g) = \cup_{x\in M}\mathcal{C}\ell(T_x^*M,g)$, where $\forall x \in M$, $\mathcal{C}\ell(T_x^*M,g) = \mathbb{R}_{1,3}$, the so called *spacetime algebra* [25]. Recall also that $\mathcal{C}\ell(M,g)$ is a vector bundle associated to the *orthonormal frame bundle*, i.e., $\mathcal{C}\ell(M,g) = P_{\mathrm{SO}^e_{(1,3)}}(M) \times_{\mathrm{Ad}} \mathcal{C}l_{1,3}$ [12, 14]. For any $x \in M$, $\mathcal{C}\ell(T_x^*M,g)$ as a linear space over the real field $\mathbb{R}$ is isomorphic to the Cartan algebra $\bigwedge(T_x^*M)$ of the cotangent space. $\bigwedge(T_x^*M) = \oplus_{k=0}^4 \bigwedge^k(T_x^*M)$, where $\bigwedge^k(T_x^*M)$ is the $\binom{4}{k}$-dimensional space of k-forms. Then, sections of $\mathcal{C}\ell(M,g)$ can be represented as a sum of non homogeneous differential forms. Let $\{e_{\mathbf{a}}\} \in \sec P_{\mathrm{SO}^e_{(1,3)}}(M)$ (the frame bundle) be an orthonormal basis for $TU \subset TM$, i.e., $\mathbf{g}(e_{\mathbf{a}}, e_{\mathbf{a}}) = \eta_{\mathbf{ab}} = \mathrm{diag}(1,-1,-1,-1)$. Let $\theta^{\mathbf{a}} \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ ($\mathbf{a} = 0,1,2,3$) be such that the set $\{\theta^{\mathbf{a}}\}$ is the dual basis of $\{e_{\mathbf{a}}\}$.

2.1 Clifford Product

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by $\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \theta^{\mathbf{b}}\theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}}$ and if $\mathcal{C} \in \sec \mathcal{C}\ell(M,g)$ we have

$$\mathcal{C} = s + v_{\mathbf{a}}\theta^{\mathbf{a}} + \frac{1}{2!}f_{\mathbf{ab}}\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \frac{1}{3!}t_{\mathbf{abc}}\theta^{\mathbf{a}}\theta^{\mathbf{b}}\theta^{\mathbf{c}} + p\theta^5 \ , \tag{1}$$

where $\tau_{\mathbf{g}} = \theta^5 = \theta^0\theta^1\theta^2\theta^3$ is the volume element and s, $v_{\mathbf{a}}$, $f_{\mathbf{ab}}$, $t_{\mathbf{abc}}$, $p \in \sec \bigwedge^0(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$.

For $A_r \in \sec \bigwedge^r(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g), B_s \in \sec \bigwedge^s(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ we define the *exterior product* in $\mathcal{C}\ell(M,g)$ $(\forall r,s = 0,1,2,3)$ by

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r, \end{aligned} \tag{2}$$

where $\langle \ \ \rangle_k$ is the component in $\bigwedge^k(T^*M)$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(M,g)$.

Let $A_r \in \sec \bigwedge^r(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g), B_s \in \sec \bigwedge^s(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$. We define a *scalar product* in $\mathcal{C}\ell(M,g)$ (denoted by $\cdot$) as follows:

(i) For $a, b \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$,

$$a \cdot b = \frac{1}{2}(ab + ba) = g(a,b). \tag{3}$$

(ii) For $A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r$,

$$\begin{aligned} A_r \cdot B_r &= (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) \\ &= \begin{vmatrix} a_1 \cdot b_1 & .... & a_1 \cdot b_r \\ .... & .... & .... \\ a_r \cdot b_1 & .... & a_r \cdot b_r \end{vmatrix} \end{aligned} \tag{4}$$

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, then $A_r \cdot B_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\mathcal{C}\ell(M, g)$.

For $r \leq s, A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s$ we define the *left contraction* by

$$\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s = \sum_{i_1 < ... < i_r} \epsilon^{i_1 ... i_s} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r})^{\sim} b_{i_r + 1} \wedge ... \wedge b_{i_s} \tag{5}$$

where $\sim$ is the reverse mapping (*reversion*) defined by

$$\sim : \sec \bigwedge\nolimits^p (T^* M) \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1 \tag{6}$$

and extended by linearity to all sections of $\mathcal{C}\ell(M, g)$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 (T^* M)$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 (T^* M)$, $A_r \in \sec \bigwedge^r (T^* M), B_s \in \sec \bigwedge^s (M)$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}\ell(M, g)$, i.e., for $A, B \in \sec \mathcal{C}\ell(M, g)$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, r \leq s \tag{7}$$

It is also necessary to introduce the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r \in \sec \bigwedge^r (T^* M), B_s \in \sec \bigwedge^s (T^* M)$ then $A_r \llcorner (\alpha B_s) = (\alpha A_r) \llcorner B_s$. See also the third formula in Eq.(8).

The main formulas used in the Clifford calculus can be obtained from the following ones (where $a \in \sec \bigwedge^1 (T^* M) \hookrightarrow \sec \mathcal{C}\ell(M, g)$):

$$\begin{aligned}
aB_s &= a \lrcorner B_s + a \wedge B_s, \;\; B_s a = B_s \llcorner a + B_s \wedge a, \\
a \lrcorner B_s &= \frac{1}{2}(aB_s - (-)^s B_s a), \\
A_r \lrcorner B_s &= (-)^{r(s-1)} B_s \llcorner A_r, \\
a \wedge B_s &= \frac{1}{2}(aB_s + (-)^s B_s a), \\
A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + ... + \langle A_r B_s \rangle_{|r+s|} \\
&= \sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}, \\
A_r \cdot B_r &= B_r \cdot A_r = \tilde{A}_r \lrcorner B_r = A_r \llcorner \tilde{B}_r = \langle \tilde{A}_r B_r \rangle_0 = \langle A_r \tilde{B}_r \rangle_0.
\end{aligned} \tag{8}$$

2.1.1 Hodge Star Operator

Let $\star$ be the Hodge star operator, i.e., the mapping

$$\star : \bigwedge\nolimits^k (T^* M) \to \bigwedge\nolimits^{4-k} (T^* M), \;\; A_k \mapsto \star A_k$$

where for $A_k \in \sec \bigwedge^k(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$

$$[B_k \cdot A_k]\tau_g = B_k \wedge \star A_k, \forall B_k \in \sec \bigwedge^k(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g). \tag{9}$$

$\tau_{\mathbf{g}} \in \bigwedge^4(M)$ is a *standard* volume element. Then we can verify that

$$\star A_k = \widetilde{A}_k \gamma^5. \tag{10}$$

2.1.2 Dirac Operator

Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\mathcal{C}\ell(M,g)$. If $A_p \in \sec \bigwedge^p(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$, then $\delta A_p = (-)^p \star^{-1} d \star A_p$, with $\star^{-1}\star =$ identity.

The Dirac operator acting on sections of $\mathcal{C}\ell(M,g)$ is the invariant first order differential operator

$$\boldsymbol{\partial} = \theta^{\mathbf{a}} \nabla_{e_{\mathbf{a}}}, \tag{11}$$

where $\{e_{\mathbf{a}}\}$ is an arbitrary *orthonormal basis* for $TU \subset TM$ and $\{\theta^{\mathbf{b}}\}$ is a basis for $T^*U \subset T^*M$ dual to the basis $\{e_{\mathbf{a}}\}$, i.e., $\theta^{\mathbf{b}}(e_{\mathbf{a}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$, $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$. The reciprocal basis of $\{\theta^{\mathbf{b}}\}$ is denoted $\{\theta_{\mathbf{a}}\}$ and we have $\theta_{\mathbf{a}} \cdot \theta_{\mathbf{b}} = \eta_{\mathbf{ab}}$. Also,

$$\nabla_{e_{\mathbf{a}}} \theta^{\mathbf{b}} = -\omega^{\mathbf{bc}}_{\mathbf{a}} \theta_{\mathbf{c}} \tag{12}$$

Defining

$$\omega_{e_{\mathbf{a}}} = \frac{1}{2} \omega^{\mathbf{bc}}_{\mathbf{a}} \theta_{\mathbf{b}} \wedge \theta_{\mathbf{c}}, \tag{13}$$

we have that for any $A_p \in \sec \bigwedge^p(T^*M)$, $p = 0, 1, 2, 3, 4$

$$\nabla_{e_{\mathbf{a}}} A_p = \partial_{e_{\mathbf{a}}} A_p + \frac{1}{2}[\omega_{e_{\mathbf{a}}}, A_p], \tag{14}$$

where $\partial_{e_{\mathbf{a}}}$ is the Pfaff derivative, i.e., if $A_p = \frac{1}{p!} A_{\mathbf{i}_1 \ldots \mathbf{i}_p} \theta^{\mathbf{i}_1 \ldots \mathbf{i}_p}$,

$$\partial_{e_{\mathbf{a}}} A_p := \frac{1}{p!} e_{\mathbf{a}}(A_{\mathbf{i}_1 \ldots \mathbf{i}_p}) \theta^{\mathbf{i}_1 \ldots \mathbf{i}_p}. \tag{15}$$

This important formula (Eq.(14) that is valid also for a nonhomogeneous $A \in \sec \mathcal{C}\ell(M,g)$ will be proved below (Section 3).

Using Eq.(14) we can show a very important result: The Dirac operator associated to a Levi-Civita connection satisfies

$$\begin{aligned} \boldsymbol{\partial} A_p &= \boldsymbol{\partial} \wedge A_p + \boldsymbol{\partial} \lrcorner A_p = dA_p - \delta A_p, \\ \boldsymbol{\partial} \wedge A_p &= dA_p, \quad \boldsymbol{\partial} \lrcorner A_p = -\delta A_p. \end{aligned} \tag{16}$$

With these results, Maxwell equations for $F \in \sec \bigwedge^2(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$, $J \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ reads

$$dF = 0, \;\; \delta F = -J, \tag{17}$$

or Maxwell equation[4] reads

$$\boldsymbol{\partial} F = J. \tag{18}$$

[4] No misprint here.

2.2 Spinor Fields

How to represent the Dirac spinor fields in this formalism? We can show that *Dirac-Hestenes* spinor fields do the job[5]. To introduce this concept in a meaningful way we need to recall several mathematical concepts. First we recall the concept of spin structure of an oriented (and time-oriented) $(M, \mathtt{g})$. This consists of a principal fibre bundle $\pi_s : \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \to M$ (called the *spin frame bundle*) with group $\mathrm{Spin}^e_{1,3}$ and a map

$$s : P_{\mathrm{Spin}^e_{1,3}}(M) \to \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M) \tag{19}$$

satisfying the following conditions:

(i) $\pi(s(p)) = \pi_s(p)\ \forall p \in \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M), \pi$ is the projection map of the bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$.

(ii) $s(pu) = s(p)\mathrm{Ad}_u$, $\forall p \in \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ and

$$\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3}),$$
$$\mathrm{Ad}_u : \mathbb{R}_{1,3} \ni x \mapsto uxu^{-1} \in \mathbb{R}_{1,3}.$$

When does a *spin structure* exist on an oriented (and time oriented) $(M, \mathtt{g})$? For a Lorentzian manifold the answer is given by a famous result due to Geroch [8] which says that for an oriented (and time-oriented) Lorentz manifold $(M, \mathbf{g})$, a spin structure exists if and only if $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ is a trivial bundle.

We call global sections $\xi \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ Lorentz frames and global sections ${}_s\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ *spin frames*. We recall (see, e.g., [14]) that each $\xi \in \sec P_{\mathrm{SO}^e_{1,3}}(M)$ is a basis for TM, which is completely specified once we give an element of the Lorentz group for each $x \in M$ and fix a fiducial frame. Each ${}_s\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ is also a basis for TM and is completely identified once we give an element of the $\mathrm{Spin}^e_{1,3}$ for each $x \in M$ and fix a fiducial frame. Note that two ordered basis for TM when considered as spin frames, even if consisting of the same vector fields, but related by a 2π rotation are considered different. Also, two ordered basis for TM when considered as spin frames, if consisting of the same vector fields, related by a 4π rotation are considered equal. Even if this mathematical construction seems at first sight impossible of experimental detection, Aharonov and Susskind [1] warrant that with clever experiments the spinor structure can be detected.

Recall that a principal bundle is trivial if and only if it admits a global section. Therefore, Geroch's result says that a (non-compact) spacetime admits a spin structure, if and only if, it admits a (globally defined) Lorentz frame. In fact, it is possible to replace $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ by $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ in the statement of Geroch theorem (see [8], footnote 25). In this way, when a (non-compact) spacetime admits a spin structure, the bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ is trivial and, therefore, every bundle associated to it is also trivial. This result is indeed a very important one, because it says to us that the real spacetime of our universe

[5] More details on other kinds of spinor fields in this formalism can be found in [20].

(that, of course, is inhabited by several different types of spinor fields) must have a topology that admits a global tetrad field, which is defined only modulus a local Lorentz transformation. The dual cotetrad have been associated to the gravitational field in [26], where we wrote wave equations for them. In a certain sense that cotetrad field is the substance of physical spacetime. In what follows we shall use the symbol Ξ to denote a spin coframe dual to a spin frame ${}_s\Xi$. We also write by *abuse of notation* that $\Xi \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$.

An oriented manifold endowed with a spin structure will be called a *spin manifold*.

2.3 Spinor Bundles and Spinor Fields

We now present the most usual definitions of spinor bundles appearing in the literature[6] and next we find appropriate vector bundles such that particular sections are *Dirac-Hestenes spinor fields.*

A *real spinor bundle* for M is a vector bundle

$$S(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbf{M} \tag{20}$$

where $\mathbf{M}$ is a left module for $\mathbb{R}_{1,3}$ and μ_l is a representation of $\mathrm{Spin}^e_{1,3}$ on $\mathrm{End}(\mathbf{M})$ given by left multiplication by elements of $\mathrm{Spin}^e_{1,3}$.

The *dual bundle* $S^\star(M)$ is a real spinor bundle

$$S^\star(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_r} \mathbf{M}^\star \tag{21}$$

where $\mathbf{M}^\star$ is a right module for $\mathbb{R}_{1,3}$ and μ_r is a representation $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M})$ given by right multiplication by (inverse) elements of $\mathrm{Spin}^e_{1,3}$. By *right multiplication* we mean that given $a \in \mathbf{M}^*$, $\mu_r(u)a = au^{-1}$, then

$$\mu_r(uu')a = a(uu')^{-1} = au'^{-1}u^{-1} = \mu_r(u)\mu_r(u')a. \tag{22}$$

A *complex spinor bundle* for M is a vector bundle

$$S_c(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}_c \tag{23}$$

where $\mathbf{M}_c$ is a complex left module for $\mathbb{C}\otimes\mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M}_c)$ given by left multiplication by elements of $\mathrm{Spin}^e_{1,3}$.

The *dual complex spinor bundle* for M is a vector bundle

$$S_c^\star(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}_c^\star \tag{24}$$

where $\mathbf{M}_c^\star$ is a complex right module for $\mathbb{C}\otimes\mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M}_c)$ given by right multiplication by (inverse) elements of $\mathrm{Spin}^e_{1,3}$.

[6] We recall that there are some other (equivalent) definitions of spinor bundles that we are not going to introduce in this paper as, e.g., the one given in [2] in terms of mappings from $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}$ to some appropriate vector space.

Taking, e.g., $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Spin}^e_{1,3} \cong Sl(2,\mathbb{C})$ in $\mathrm{End}(\mathbb{C}^4)$, we immediately recognize the usual definition of the (Dirac) covariant spinor bundle of M, as given, e.g., in [3].

Let $\{\mathbf{E}_a\}$ be an orthonormal basis $\mathbb{R}_{1,3}$. The ideal $I = \mathbb{R}_{1,3}\frac{1}{2}(1+\mathbf{E}_0)$ is a minimal left ideal of $\mathbb{R}_{1,3}$. Besides I, other ideals exist in $\mathbb{R}_{1,3}$, that are only *algebraically* equivalent to this one [25]. In order to capture all possibilities we recall that $\mathbb{R}_{1,3}$ can be considered as a module over itself by left (or right) multiplication. We have:

The *left real spin-Clifford bundle* of M is the vector bundle

$$\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_l \mathbb{R}_{1,3} \tag{25}$$

where l is the representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$. Sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ are *called left spin-Clifford fields.*

$\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ is a 'principal $\mathbb{R}_{1,3}$-bundle', i.e., it admits a free action of $\mathbb{R}_{1,3}$ on the right [12], which is denoted by R_g, $g \in \mathbb{R}_{1,3}$.

There is a *natural* embedding $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \hookrightarrow \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ which comes from the embedding $\mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}^0_{1,3}$. Hence (as we shall see in more details below), every real left spinor bundle for M can be captured from $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, which is a vector bundle very different from $\mathcal{C}\ell(M,g)$. Their relation is presented below, but before, we introduce ideal left algebraic spinor fields.

Let $I(M,g)$ be a subbundle of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ such that there exists a primitive idempotent $\mathbf{e}$ of $\mathbb{R}_{1,3}$ with

$$R_{\mathbf{e}}\Psi = \Psi \tag{26}$$

for all $\Psi \in \sec I(M,g) \hookrightarrow \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$. Then, $I(M,g)$ is called a subbundle of *ideal left algebraic spinor fields.* Any $\Psi \in \sec I(M,g) \hookrightarrow \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ is called a *left ideal algebraic spinor field* (*LIASF).*

$I(M,g)$ can be thought of as being a real spinor bundle for M such that $\mathbf{M}$ in Eq.(23) is a minimal left ideal of $\mathbb{R}_{1,3}$.

We say that two subbundles $I(M,g)$ and $I'(M,g)$ of *LIASF are geometrically equivalent if the idempotents* $\mathbf{e}, \mathbf{e}' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \mathrm{Spin}^e_{1,3}$, i.e., $\mathbf{e}' = u\mathbf{e}u^{-1}$.

The *right real spin-Clifford bundle* of M is the vector bundle

$$\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_r \mathbb{R}_{1,3}. \tag{27}$$

Sections of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ are called *right spin-Clifford fields*

In Eq.(27) r refers to a representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$, given by $r(a)x = xa^{-1}$. As in the case for the left real spin-Clifford bundle, there is a *natural* embedding $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \hookrightarrow \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ which comes from the embedding $\mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}^0_{1,3}$. There exists also a natural left L_a action of $a \in \mathbb{R}_{1,3}$ on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$. This will be proved below.

Let $I^{\star}(M,g)$ be a subbundle of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ such that there exists a primitive idempotent element $\mathbf{e}$ of $\mathbb{R}_{1,3}$ with

$$L_{\mathbf{e}}\Psi = \Psi \tag{28}$$

for any $\Psi \in \sec I^{\star}(M,g) \hookrightarrow \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$. Then, $I^{\star}(M,g)$ is called a subbundle of right *ideal algebraic spinor fields*. Any $\Psi \in \sec I^{\star}(M,g) \hookrightarrow \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ is called a RIASF. $I^{\star}(M,g)$ can be thought of as being a real spinor bundle for M such that $\mathbf{M}^{\star}$ in Eq.(24) is a minimal right ideal of $\mathbb{R}_{1,3}$.

We say that two subbundles $I^{*}(M,g)$ and $I^{*\prime}(M,g)$ of *RIASF* are *geometrically equivalent* if the idempotents $\mathbf{e},\mathbf{e}' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \mathrm{Spin}^{e}_{1,3}$, i.e., $\mathbf{e}' = u\mathbf{e}u^{-1}$.

The following result proved in [14] is crucial: in a spin manifold, we have

$$\mathcal{C}\ell(M,g) = \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}. \tag{29}$$

We recall also that $S(M,g)$ (or $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$) is a bundle of (left) *modules* over the bundle of algebras $\mathcal{C}\ell(M,g)$. In particular, the sections of the spinor bundle ($S(M,g)$ or $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$) are a module over the sections of the Clifford bundle [12]. Then, if $\Phi,\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ and $\Psi \neq 0$, there exists $\psi \in \sec \mathcal{C}\ell(M,g)$ such that

$$\Psi = \psi\Phi. \tag{30}$$

This allows to identify a *correspondence* between some sections of $\mathcal{C}\ell(M,g)$ and some sections of $I(M)$ or $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ once we fix a section on $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$. This and other correspondences are essential for the theory of Dirac-Hestenes spinor fields (more details are given in [14]). Once we clarified which is the meaning of a bundle of modules $S(M,g)$ over a bundle of algebras $\mathcal{C}\ell(M,g)$, we can say that:

Two real left spinor bundles are equivalent, if and only if, they are equivalent as bundles of $\mathcal{C}\ell(M,g)$ modules.

Of course, geometrically equivalently real left spinor bundles are equivalent.

Remark 1 *In what follows we denote the complexified left spin Clifford bundle and the complexified right spin-Clifford bundle by*

$$\begin{aligned}
\mathbb{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M) &= \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{l} \mathbb{C}\otimes\mathbb{R}_{1,3} \simeq \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{l} \mathbb{R}_{4,1},\\
\mathbb{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M) &= \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{r} \mathbb{C}\otimes\mathbb{R}_{1,3} \simeq \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{r} \mathbb{R}_{4,1}.
\end{aligned} \tag{31}$$

2.4 Dirac-Hestenes Spinor Fields

Let $\mathbf{E}^{\mu}$, $\mu = 0,1,2,3$ be the canonical basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ which generates the algebra $\mathbb{R}_{1,3}$. They satisfy the basic relation $\mathbf{E}^{\mu}\mathbf{E}^{\nu} + \mathbf{E}^{\nu}\mathbf{E}^{\mu} = 2\eta^{\mu\nu}$. We recall that [25]

$$\mathbf{e} = \frac{1}{2}(1+\mathbf{E}^{0}) \in \mathbb{R}_{1,3} \tag{32}$$

is a primitive idempotent of $\mathbb{R}_{1,3}$ and

$$\mathbf{f} = \frac{1}{2}(1+\mathbf{E}^0)\frac{1}{2}(1+i\mathbf{E}^2\mathbf{E}^1) \in \mathbb{C}\otimes\mathbb{R}_{1,3} \tag{33}$$

is a primitive idempotent of $\mathbb{C}\otimes\mathbb{R}_{1,3}$. Now, let $\mathbf{I} = \mathbb{R}_{1,3}\mathbf{e}$ and $\mathbf{I}_{\mathbb{C}} = \mathbb{C}\otimes\mathbb{R}_{1,3}\mathbf{f}$ be respectively the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C}\otimes\mathbb{R}_{1,3}$ generated by $\mathbf{e}$ and $\mathbf{f}$. Let $\phi = \phi\mathbf{e} \in \mathbf{I}$ and $\mathbf{\Psi} = \mathbf{\Psi f} \in \mathbf{I}_{\mathbb{C}}$. Then, any $\phi \in \mathbf{I}$ can be written as

$$\phi = \psi\mathbf{e} \tag{34}$$

with $\psi \in \mathbb{R}^0_{1,3}$. Analogously, any $\mathbf{\Psi} \in \mathbf{I}_{\mathbb{C}}$ can be written as

$$\mathbf{\Psi} = \psi\mathbf{e}\frac{1}{2}(1+i\mathbf{E}^2\mathbf{E}^1), \tag{35}$$

with $\psi \in \mathbb{R}^0_{1,3}$.

Recall moreover that $\mathbb{C}\otimes\mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, where $\mathbb{C}(4)$ is the algebra of the 4×4 complex matrices. We can verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{36}$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of $\mathbf{f}$. In that way (see details in [25]) there is a bijection between column spinors, i.e., elements of $\mathbb{C}^4$ (the complex 4-dimensional vector space) and the elements of $\mathbf{I}_{\mathbb{C}}$. All that, plus the definitions of the left real and complex spin bundles and the subbundle $I(M,g)$ suggests the following definition [14]:

Let $\Phi \in \sec I(M,\mathsf{g}) \hookrightarrow \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g})$, i.e.,

$$R_{\mathbf{e}}\Phi = \Phi\mathbf{e} = \Phi,\ \mathbf{e}^2 = \mathbf{e} = \mathbf{\frac{1}{2}}(\mathbf{1}+\mathbf{E^0}) \in \mathbb{R}_{1,3}. \tag{37}$$

A *Dirac-Hestenes spinor field* (*DHSF*) associated with Φ is an *even* section ψ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ such that

$$\Phi = \psi\mathbf{e}. \tag{38}$$

An equivalent definition of a *DHSF* is the following. Let $\Psi \in \sec \mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ be such that

$$R_{\mathbf{f}}\Psi = \Psi\mathbf{f} = \Psi, \mathbf{f}^2 = \mathbf{f} = \mathbf{\frac{1}{2}}(\mathbf{1}+\mathbf{E^0})\mathbf{\frac{1}{2}}(\mathbf{1}+\mathbf{iE^2E^1}) \in \mathbb{C}\otimes\mathbb{R}_{1,3}. \tag{39}$$

Then, a *DHSF* associated to Ψ is an even section ψ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g}) \hookrightarrow \mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g})$ such that

$$\Psi = \psi\mathbf{f}. \tag{40}$$

In what follows, when we refer to a *DHSF* ψ we omit for simplicity the wording associated with Φ (or Ψ). It is very important to observe that *DHSF* are *not* sums of even multivector fields although, under a local trivialization, $\psi \in \sec C\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$ is mapped on an even element[7] of $\mathbb{R}_{1,3}$. We emphasize that *DHSF* are particular sections of a spinor bundle, not of the Clifford bundle. However, we show below how these objects have *representatives* in the Clifford bundle. To understand how this happens, recall that, when M is a spin manifold:

(i) The elements of $\mathcal{C}\ell(M,\mathtt{g}) = \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$ are equivalence classes $[(p,a)]$ of pairs (p,a), where $p \in \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = uau^{-1}$, for some $u \in \mathrm{Spin}^{e}_{1,3}$;

(ii) The elements of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$ are equivalence classes of pairs (p,a), where $p \in \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = ua$, for some $u \in \mathrm{Spin}^{e}_{1,3}$;

(iii) The elements of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ are equivalence classes of pairs (p,a), where $p \in \mathbf{P}_{\mathrm{Spin}^{e}_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p,a) \sim (p',a') \Leftrightarrow p' = pu^{-1}$, $a' = au^{-1}$, for some $u \in \mathrm{Spin}^{e}_{1,3}$.

In this way, it is possible to define the following natural actions on these associated bundles [14]:

1. There is a natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$ and a natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$.

Indeed, given $b \in \mathbb{R}_{1,3}$ and $\alpha \in \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$, select a representative (p,a) for α and define $\alpha b := [(p,ab)]$. If another representative (pu^{-1},ua) is chosen for α, we have $(pu^{-1},uab) \sim (p,ab)$ and thus αb is a well-defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$.

Let us denote the space of $\mathbb{R}_{1,3}$-valued smooth functions on M by $\mathcal{F}(M,\mathbb{R}_{1,3})$. Then, the above proposition immediately yields the following.

2. There is a natural right action of $\mathcal{F}(M,\mathbb{R}_{1,3})$ on the sections of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$ and a natural left action of $\mathcal{F}(M,\mathbb{R}_{1,3})$ on the sections of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$.

3. There is a natural left action of $\sec\mathcal{C}\ell(M,\mathtt{g})$ on the sections of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$ and a natural right action of $\sec\mathcal{C}\ell(M,\mathtt{g})$ on sections of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$.

Indeed, given $\alpha \in \sec\mathcal{C}\ell(M,\mathtt{g})$ and $\beta \in \sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p,ab)] \in \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathtt{g})$. If alternative representatives (pu^{-1},uau^{-1}) and (pu^{-1},ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have

$$(pu^{-1},uau^{-1}ub) = (pu^{-1},uab) \sim (p,ab)$$

and thus $(\alpha\beta)(x)$ is a well-defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$.

[7] Note that it is meaningful to speak about even (or odd) elements in $C\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ since $\mathrm{Spin}^{e}_{1,3} \hookrightarrow \mathbb{R}^{0}_{1,3}$.

4. There is a natural pairing

$$\sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g) \times \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g) \to \sec \mathcal{C}\ell(M,g).$$

Indeed, given $\alpha \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\beta \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p,ab)] \in \mathcal{C}\ell(M,\mathsf{g})$. If alternative representatives (pu^{-1}, ua) and (pu^{-1}, bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$, we have $(pu^{-1}, uabu^{-1}) \sim (p,ab)$ and thus $(\alpha\beta)(x)$ is a well-defined element of $\mathcal{C}\ell(M,g)$.

5. There is a natural pairing

$$\sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g}) \times \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g}) \to \mathcal{F}(M,\mathbb{R}_{1,3}).$$

Indeed, given $\alpha \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g})$ and $\beta \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathsf{g})$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := ab \in \mathbb{R}_{1,3}$. If alternative representatives (pu^{-1}, au^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have $au^{-1}ub = ab$ and thus $(\alpha\beta)(x)$ is a well-defined element of $\mathbb{R}_{1,3}$.

2.4.1 "Unit Sections"

We now show how to define "unit sections" on the various vector bundles associated to the principal bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$.

Let

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathrm{Spin}^e_{1,3}, \quad \Phi_j : \pi^{-1}(U_j) \to U_j \times \mathrm{Spin}^e_{1,3}$$

be two local trivializations for $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$, with

$$\Phi_i(u) = (\pi(u) = x, \phi_{i,x}(u)), \quad \Phi_j(u) = (\pi(u) = x, \phi_{j,x}(u)).$$

Recall that the transition function on $g_{ij} : U_i \cap U_j \to \mathrm{Spin}^e_{1,3}$ is then given by

$$g_{ij}(x) = \phi_{i,x}(u) \circ \phi_{j,x}(u)^{-1},$$

which does not depend on u.

6. $\mathcal{C}\ell(M,g)$ has a naturally defined global unit section. Indeed, for the associated bundle $\mathcal{C}\ell(M,g)$, the transition functions corresponding to local trivializations

$$\Psi_i : \pi_c^{-1}(U_i) \to U_i \times \mathbb{R}_{1,3}, \quad \Psi_j : \pi_c^{-1}(U_j) \to U_j \times \mathbb{R}_{1,3}, \tag{41}$$

are given by $h_{ij}(x) = \mathrm{Ad}_{g_{ij}(x)}$. Define the local sections

$$\mathbf{1}_i(x) = \Psi_i^{-1}(x,1), \quad \mathbf{1}_j(x) = \Psi_j^{-1}(x,1), \tag{42}$$

where 1 is the unit element of $\mathbb{R}_{1,3}$. Since

$$h_{ij}(x) \cdot 1 = \mathrm{Ad}_{g_{ij}(x)}(1) = g_{ij}(x) 1 g_{ij}(x)^{-1} = 1,$$

we see that the expressions above uniquely define a global section $\mathbf{1} \in \mathcal{C}\ell(M,g)$ with $\mathbf{1}|_{U_i} = \mathbf{1}_i$.

7. There exists a unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,g)$ (and also on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,g)$), if and only if, $\mathbf{P}_{\mathrm{Spin}^e_{p,q}}(M)$ is trivial.

Indeed, we show the necessity for the case of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,g)$,[8] the sufficiency is trivial. For $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,g)$, the transition functions corresponding to local trivializations

$$\Omega_i : \pi_{sc}^{-1}(U_i) \to U_i \times \mathbb{R}_{p,q}, \quad \Omega_j : \pi_{sc}^{-1}(U_j) \to U_j \times \mathbb{R}_{p,q}, \tag{43}$$

are given by $k_{ij}(x) = R_{g_{ij}(x)}$, with $R_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, x \mapsto xa^{-1}$. A unit section in $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$ – if it exists – is written in terms of these two local trivializations as

$$\mathbf{1}^r_i(x) = \Omega_i^{-1}(x,1), \quad \mathbf{1}^r_j(x) = \Omega_j^{-1}(x,1), \tag{44}$$

and we must have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x)\ \forall x \in U_i \cap U_j$. As $\Omega_i(\mathbf{1}^r_i(x)) = (x,1) = \Omega_j(\mathbf{1}^r_j(x))$, we have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x) \Leftrightarrow 1 = k_{ij}(x)\cdot 1 \Leftrightarrow 1 = 1g_{ij}(x)^{-1} \Leftrightarrow g_{ij}(x) = 1$.[9]

We already recalled that when the (non-compact) spacetime M is a spin manifold, the bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ admits global sections. With this in mind, let us fix a *spin frame* ${}_s\Xi$ and its dual spin *coframe* Ξ for M. This induces a global trivialization for $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$, which we denote by $\Phi_\Xi : \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \to M \times \mathrm{Spin}^e_{1,3}$, with $\Phi_\Xi^{-1}(x,1) = \Xi(x)$. As we show in the following, the spin coframe Ξ can also be used to induce certain fiducial global sections on the various vector bundles associated to $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$:

(a) $\mathcal{C}\ell(M,g)$

Let $\{\mathbf{E}^{\mathbf{a}}\}$ be a fixed orthonormal basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ (which can be thought of as being the *canonical* basis of $\mathbb{R}^{1,3}$). We define basis sections in $\mathcal{C}\ell(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$ by $\theta_{\mathbf{a}}(x) = [(\Xi(x), \mathbf{E}_a)]$. Of course, this induces a *multiform* basis $\{\boldsymbol{\theta}_I(x)\}$ for each $x \in M$. Note that a more precise notation for $\theta_{\mathbf{a}}$ would be, for instance, $\theta^{(\Xi)}_{\mathbf{a}}$.

(b) $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$

Let $\mathbf{1}^l_\Xi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ be defined by $\mathbf{1}^l_\Xi(x) = [(\Xi(x),1)]$. Then the natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ leads to $\mathbf{1}^l_\Xi(x)a = [(\Xi(x),a)]$

[8] The proof for the case of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,g)$ is analogous.

[9] For general spin manifolds, the bundle $\mathbf{P}_{\mathrm{Spin}^e_{p,q}}(M)$ is not necessarily trivial for arbitrary (p,q), but Geroch's theorem warrants that, for the special case $(p,q) = (1,3)$ with M non-compact, $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ is trivial. By the above proposition, we then see that $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ and also $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,g)$ have global "unit sections". It is most important to note, however, that each different choice of a (global) trivialization Ω_i on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ (respectively $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,g)$) induces a different global unit section $\mathbf{1}^r_i$ (respectively $\mathbf{1}^l_i$). Therefore, even in this case there is no canonical unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ (respectively on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$).

for all $a \in \mathbb{R}_{1,3}$. It follows from property **2** above that an arbitrary section $\alpha \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$ can be written as $\alpha = \mathbf{1}^{l}_{\Xi} f$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

(c) $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$

Let $\mathbf{1}^{r}_{\Xi} \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$ be defined by $\mathbf{1}^{r}_{\Xi}(x) = [(\Xi(x), 1)]$. Then the natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$ leads to $a\mathbf{1}^{r}_{\Xi}(x) = [(\Xi(x), a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from property **2** that an arbitrary section $\alpha \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$ can be written as $\alpha = f\mathbf{1}^{r}_{\Xi}$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

Now recall again that a spin structure on M is a 2-1 bundle map $s : \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \to \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ such that $s(pu) = s(p)\mathrm{Ad}_u$, $\forall p \in P_{\mathrm{Spin}^e_{1,3}}(M)$, $u \in \mathrm{Spin}^e_{1,3}$, where $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}$, $\mathrm{Ad}_u : x \mapsto uxu^{-1}$. We see that the specification of the global section in the case (**a**) above is compatible with the Lorentz coframe $\{\theta_a\} = s(\Xi)$ assigned by s. More precisely, for each $x \in M$, the element $s(\Xi(x)) \in \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ is to be regarded as a proper isometry $s(\Xi(x)) : \mathbb{R}^{1,3} \to T^*_x M$, so that $\theta_{\mathbf{a}}(x) := s(p) \cdot \mathbf{E}_{\mathbf{a}}$ yields a Lorentz coframe $\{\theta_{\mathbf{a}}\}$ on M, which we denoted by $s(\Xi)$. On the other hand, $\mathcal{C}\ell(M,g)$ is isomorphic to $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$, and we can always arrange things so that $\theta_{\mathbf{a}}(x)$ is represented in this bundle as $\theta_{\mathbf{a}}(x) = [(\Xi(x), \mathbf{E}_{\mathbf{a}})]$. In fact, all we have to do is to verify that this identification is covariant under a change of coframes. To see that, let $\Xi' \in \sec(\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M))$ be another spin coframe on M. From the principal bundle structure of $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$, we know that, for each $x \in M$, there exists (a unique) $u(x) \in \mathrm{Spin}^e_{1,3}$ such that $\Xi'(x) = \Xi(x)u(x)$. If we define, as above, $\theta'_{\mathbf{a}}(x) = s(\Xi'(x)) \cdot \mathbf{E}_{\mathbf{a}}$, then

$$\begin{aligned}
\theta'_{\mathbf{a}}(x) &= s(\Xi(x)u(x)) \cdot \mathbf{E}_{\mathbf{a}} = s(\Xi(x))\mathrm{Ad}_{u(x)} \cdot \mathbf{E}_{\mathbf{a}} \\
&= \left[(\Xi(x), \mathrm{Ad}_{u(x)} \cdot \mathbf{E}_{\mathbf{a}})\right] = [(\Xi(x)u(x), \mathbf{E}_{\mathbf{a}})] = [(\Xi'(x), \mathbf{E}_a)]\,.
\end{aligned}$$

This proves our claim. The following results proved in [14] are also needed for what follows.

$$\begin{aligned}
\text{(i) } & \mathbf{E}_{\mathbf{a}} = \mathbf{1}^{r}_{\Xi}(x)\theta_{\mathbf{a}}(x)\mathbf{1}^{l}_{\Xi}(x),\ \forall x \in M, \\
\text{(ii) } & \mathbf{1}^{l}_{\Xi}\mathbf{1}^{r}_{\Xi} = 1 \in \mathcal{C}\ell(M,g), \\
\text{(iii) } & \mathbf{1}^{r}_{\Xi}\mathbf{1}^{l}_{\Xi} = 1 \in \mathbb{R}_{1,3}.
\end{aligned} \tag{45}$$

Consider now how the various global sections defined above transform when the spin coframe Ξ is changed. Let $\Xi' \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ be another spin coframe with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \mathrm{Spin}^e_{1,3}$. Let $\theta_{\mathbf{a}}$, $\mathbf{1}^{r}_{\Xi}$, $\mathbf{1}^{l}_{\Xi}$ and $\theta'_{\mathbf{a}}$, $\mathbf{1}^{r}_{\Xi'}$, $\mathbf{1}^{l}_{\Xi'}$ be the global sections respectively defined by Ξ and Ξ' (as above). We then have (see proof in [14]) that if Ξ, Ξ' are two spin coframes related by $\Xi' = \Xi u$, where $u : M \to \mathrm{Spin}^e_{1,3}$, then

$$\begin{aligned}
\text{(i) } & \theta'_{\mathbf{a}} = U\theta_{\mathbf{a}}U^{-1} \\
\text{(ii) } & \mathbf{1}^{l}_{\Xi'} = \mathbf{1}^{l}_{\Xi}u = U\mathbf{1}^{l}_{\Xi}, \\
\text{(iii) } & \mathbf{1}^{r}_{\Xi'} = u^{-1}\mathbf{1}^{r}_{\Xi} = \mathbf{1}^{r}_{\Xi}U^{-1},
\end{aligned} \tag{46}$$

where $U \in \sec \mathcal{C}\ell(M,g)$ is the Clifford field associated to u by $U(x) = [(\Xi(x), u(x))]$. Also, in (ii) and (iii), u and u^{-1} respectively act on $\mathbf{1}_{\Xi}^{l} \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ and $\mathbf{1}_{\Xi}^{r} \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$.

3 Covariant Derivatives of Clifford Fields

Since the Clifford bundle of differential forms is $\mathcal{C}\ell(M,g) = \mathcal{T}M/J_{\mathbf{g}}$, it is clear that any linear connection ∇ on $\mathcal{T}M$ which is metric compatible ($\nabla \mathbf{g} = 0$) passes to the quotient $\mathcal{T}M/J_{\mathbf{g}}$, and thus define an algebra bundle connection [4]. In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M,g)$ is completely determined.

We now derive [14] formulas for the covariant derivative of Clifford fields (and for Dirac-Hestenes spinor fields) using the general theory of connections in principal bundles and covariant derivatives in associate vector bundles as developed in the Appendices (see also, e.g., [3, 11, 21])

The covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M,g)$ (in a given gauge), in the direction of the vector field $V \in \sec TM$ is given by

$$\nabla_V A = \partial_V(A) + \frac{1}{2}[\omega_V, A], \tag{47}$$

where ω_V is the usual ($\bigwedge^2(T^*M)$-valued) connection 1-form written in the basis $\{\theta_{\mathbf{a}}\}$. We recall that ∂_V is the Pfaff derivative operator (Eq.(15)), that is, if $A = A^I \theta_I$, then $\partial_V(A) := V(A^I)\theta_I$.[10]

Indeed, writing $A(t) = A(\sigma(t))$ in terms of the multiform basis $\{\varepsilon_I\}$ of sections associated to a given spin coframe, we have $A(t) = A^I(t)\varepsilon_I(t) = A^I(t)[(\Xi(t), \mathbf{E}_I)] = [(\Xi(t), A^I(t)\mathbf{E}_I)] = [(\Xi(t), a(t))]$, with $a(t) := A^I(t)\mathbf{E}_I \in \mathbb{R}_{1,3}$. Then the parallel transported field is given by [3]

$$A^{0}_{||t} = [(\Xi(0), g(t)a(t)g(t)^{-1})] \tag{48}$$

for some $g(t) \in \mathrm{Spin}^{e}_{1,3}$, with $g(0) = 1$. Thus

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}\left[g(t)a(t)g(t)^{-1} - a(0)\right] &= \left[\frac{dg}{dt}ag^{-1} + g\frac{da}{dt}g^{-1} + ga\frac{dg_t^{-1}}{dt}\right]_{t=0} = \\ &= \dot{a}(0) + \dot{g}(0)a(0) - a(0)\dot{g}(0) = \\ &= V(A^I)\mathbf{E}_I + [\dot{g}(0), a(0)],\end{aligned}$$

where $\dot{g}(0) \in \mathrm{spin}^{e}_{1,3} = \bigwedge^2(\mathbb{R}^{1,3})$. Therefore

$$\nabla_V A = V(A^I)\theta_I + \frac{1}{2}[\omega_V, A],$$

[10] I denotes collective indices of a basis of $\mathcal{C}\ell(M,\mathbf{g})$, e.g., $\theta_{12} = \theta_1\theta_2$.

where

$$\omega_V = [(p, \dot{g}(0))]= \frac{1}{2}\omega^{\mathbf{a}}_{V\mathbf{b}}\theta_{\mathbf{a}} \wedge \theta^{\mathbf{b}} = \frac{1}{2}\omega^{\mathbf{a}}{}_{V}{}^{\mathbf{b}}\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}} \tag{49}$$

$$= \frac{1}{2}V^{\mathbf{c}}\omega^{\mathbf{ab}}_{\mathbf{c}}\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}} = \frac{1}{2}V^{\mathbf{c}}\omega_{\mathbf{acb}}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}. \tag{50}$$

In particular, calculating the covariant derivative of the basis 1-covector fields ε_a yields $\frac{1}{2}[\omega_{\mathbf{e_c}}, \theta_{\mathbf{a}}] = \omega^{\mathbf{b}}_{\mathbf{ca}}\theta_{\mathbf{b}}$. Note that

$$\omega_{\mathbf{acb}} = \eta_{\mathbf{ad}}\omega^{\mathbf{d}}_{\mathbf{c}\ \mathbf{b}} = -\omega_{\mathbf{bca}} \tag{51}$$

and

$$\omega^{\mathbf{ab}}_{\mathbf{c}} = \eta^{\mathbf{ka}}\omega_{\mathbf{kcl}}\eta^{\mathbf{lb}} = -\omega^{\mathbf{ba}}_{\mathbf{c}} \tag{52}$$

In this way, $\omega : V \mapsto \omega_V$ is the usual ($\bigwedge^2(T^*M)$-valued) connection 1-form on M written in a given gauge (i.e., relative to a spin frame and associated orthonormal (vector) coframe).

Eq.(47) shows that the covariant derivative preserves the degree of a homogeneous Clifford field, as can be easily verified.

The general formula given by Eq.(47) and the associative law in the Clifford algebra immediately yields the result that the covariant derivative ∇_V on $\mathcal{C}\ell(M, g)$ acts as a derivation on the algebra of sections, i.e., for $A, B \in \sec \mathcal{C}\ell(M, g)$ and $V \in \sec TM$, it holds

$$\nabla_V(AB) = (\nabla_V A)B + A(\nabla_V B). \tag{53}$$

Under a change of gauge (local Lorentz transformation), ω_V transforms as

$$\frac{1}{2}\omega_V \mapsto U\frac{1}{2}\omega_V U^{-1} + (\nabla_V U)U^{-1}, \tag{54}$$

Indeed, using Eq.(47) we can calculate $\nabla_V A$ in two different gauges as

$$\nabla_V A = \partial_V(A) + \frac{1}{2}[\omega_V, A], \tag{55}$$

or

$$\nabla_V A = \partial'_V(A) + \frac{1}{2}[\omega'_V, A], \tag{56}$$

where by definition $\partial_V(A) = V(A_I)\theta^I$ and $\partial'_V(A) = V(A'_I)\theta'^I$. Now, we observe that since $\theta'^I = U\theta^I U^{-1}$, we can write

$$\begin{aligned} U\partial_V(U^{-1}A) &= U\partial_V(U^{-1}A'_I\theta'^I) = V(A'_I)\theta'^I + A'_I\theta'^I U\partial_V(U^{-1}) \\ &= \partial'_V(A) + AU\partial_V(U^{-1}). \end{aligned}$$

Now, $U\partial_V(U^{-1}A) = \partial_V A + U(\partial_V U^{-1})A$ and it follows that

$$\partial'_V(A) = \partial_V(A) - [(\partial_V U)\, U^{-1}, A]$$

Then, we see comparing the second members of Eq.(55) and Eq.(56) that

$$\frac{1}{2}\omega_V = \frac{1}{2}\omega'_V + U\left(\partial_V U^{-1}\right),$$
$$\frac{1}{2}\omega'_V = \frac{1}{2}\omega_V + \left(\partial_V U\right) U^{-1}, \tag{57}$$

and

$$\begin{aligned}\frac{1}{2}\omega'_V &= \frac{1}{2}\omega_V + \left[\nabla_V U - \frac{1}{2}\omega_V U + \frac{1}{2}U\omega_V\right] U^{-1} \\ &= \frac{1}{2}U\omega_V U^{-1} + (\nabla_V U)U^{-1}.\end{aligned} \tag{58}$$

3.1 Curvature and Torsion Extensors of a Riemann-Cartan Connection

Let $\mathbf{u},\mathbf{v},\mathbf{t},\mathbf{z} \in \sec TM$ and $u,v,t,z \in \bigwedge^1(T^*M) \hookrightarrow \mathcal{C}\ell(M,g)$ the physically equivalent 1-forms, i.e., $u = \mathbf{g}(\mathbf{u},\ )$, etc. Let moreover, as usual $\{e_{\mathbf{a}}\}$ be an orthonormal basis for TM and $\{\theta^{\mathbf{a}}\}$, $\theta^{\mathbf{a}} \in \sec\bigwedge^1(T^*M) \hookrightarrow \mathcal{C}\ell(M,g)$ the corresponding dual basis and consider the Riemann-Cartan spacetime $(M,\mathbf{g},\tau_{\mathbf{g}},\uparrow,\nabla)$. Call $\partial = \theta^{\mathbf{a}}e_{\mathbf{a}}$.

The *torsion operator* of the connection is the $(2,1)$-extensor field[11]

$$\tau : \sec\bigwedge^2(T^*M) \to \sec\bigwedge^1(T^*M),$$
$$\tau(u\wedge v) = (u\cdot\boldsymbol{\partial})v - (v\cdot\boldsymbol{\partial})u - [u,v], \tag{59}$$

where the Lie bracket of form fields is[12]

$$[u,v] = (u\cdot\partial)v - (v\cdot\partial)u, \tag{60}$$

The *torsion* of the connection is the extensor field

$$\mathbf{T}\colon \sec\left(\bigwedge^1(T^*M)\times\bigwedge^2(T^*M)\right) \to \sec\bigwedge^0(T^*M), \tag{61}$$
$$\mathbf{T}(z,u\wedge v) = z\cdot\tau(u\wedge v). \tag{62}$$

The *connection* $(1,2)$-extensor field $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} : \sec\bigwedge^1(T^*M) \to \sec\bigwedge^2(T^*M),$$
$$v \mapsto \boldsymbol{\omega}_v = \omega_{\mathbf{v}},\ v = \mathbf{g}(\mathbf{v},\ ), \tag{63}$$

where $\omega_{\mathbf{v}}$ is the ($\bigwedge^2(T^*M)$-valued) connection 1-form introduced in Eq.(49).

[11] For a detailed theory of multivector and extensor fields, see [5, 6, 7, 16, 17, 18, 19].

[12] $\mathbf{e_a}\theta_{\mathbf{b}} := h^{\mu}_{\mathbf{a}}\partial_\mu(\eta_{\mathbf{bc}}\theta^{\mathbf{c}}) = h^{\mu}_{\mathbf{a}}\eta_{\mathbf{bc}}\partial_\mu(h^{\mathbf{c}}_{\nu}dx^\nu) = h^{\mu}_{\mathbf{a}}\partial_\mu(h_{\mathbf{b}\nu})dx^\nu$

The *curvature biform* of the connection is the (2,2)-extensor field[13] $\mathfrak{R}(u\wedge v)$

$$\mathfrak{R}:\sec\bigwedge^2(T^*M)\rightarrow\sec\bigwedge^2(T^*M)$$
$$\mathfrak{R}(u\wedge v)=u\cdot\partial\boldsymbol{\omega}(v)-v\cdot\partial\boldsymbol{\omega}(u)+\boldsymbol{\omega}(u)\times\boldsymbol{\omega}(v). \tag{64}$$

We now show that the curvature biform can be written as

$$\mathfrak{R}(u\wedge v)=u\cdot\boldsymbol{\partial}\omega_{\mathbf{v}}-v\cdot\boldsymbol{\partial}\omega_{\mathbf{u}}-\frac{1}{2}[\omega_{\mathbf{u}},\omega_{\mathbf{v}}]-\omega_{[\mathbf{u},\mathbf{v}]}. \tag{65}$$

Indeed, recall that by definition

$$u\cdot\boldsymbol{\partial}(\boldsymbol{\omega}(v))=u\cdot\partial(\boldsymbol{\omega}(v))+[\boldsymbol{\omega}(u),\boldsymbol{\omega}(v)]. \tag{66}$$

Also,

$$(u\cdot\partial\boldsymbol{\omega})\,(v)\equiv u\cdot\partial\boldsymbol{\omega}(v)=u\cdot\partial(\boldsymbol{\omega}(v))-\boldsymbol{\omega}(u\cdot\partial v). \tag{67}$$

So,

$$\begin{aligned}u\cdot\partial(\boldsymbol{\omega}(v))-v\cdot\partial(\boldsymbol{\omega}(u))&=u\cdot\partial\boldsymbol{\omega}(v)-v\cdot\partial\boldsymbol{\omega}(u)+\boldsymbol{\omega}(u\cdot\partial v)-\boldsymbol{\omega}(v\cdot\partial u)\\&=u\cdot\partial\boldsymbol{\omega}(v)-v\cdot\partial\boldsymbol{\omega}(u)+\boldsymbol{\omega}([u,v])\\&=u\cdot\partial\omega_{\mathbf{v}}-v\cdot\partial\omega_{\mathbf{u}}+\omega_{[\mathbf{u},\mathbf{v}]},\end{aligned} \tag{68}$$

and using the above equations in Eq.(64) we derives Eq.(65).

We can also easily show that

$$[\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}]t-\nabla_{[\mathbf{u},\mathbf{v}]}t=\frac{1}{2}[\mathfrak{R}(u\wedge v),t] \tag{69}$$

It is important to have in mind that the curvature operator of the theory of covariant derivatives is given by (see, e.g., [3])

$$\rho(\mathbf{u},\mathbf{v})=[\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}]-\nabla_{[\mathbf{u},\mathbf{v}]} \tag{70}$$

The *Riemann curvature extensor* of the connection is

$$\mathbf{R}:\sec\bigwedge^2(T^*M)\times\bigwedge^2(T^*M)\rightarrow\sec\bigwedge^0(T^*M),$$
$$\mathbf{R}(t\wedge z,u\wedge v)=(t\wedge z)\cdot\mathfrak{R}(u\wedge v) \tag{71}$$

We can show that if $[e_{\mathbf{a}},e_{\mathbf{b}}]=c_{\mathbf{ab}}^{\mathbf{d}}e_{\mathbf{d}}$, then $[\theta_{\mathbf{a}},\theta_{\mathbf{a}}]=c_{\mathbf{ab}}^{\mathbf{d}}\theta_{\mathbf{d}}$. Then, using the above formulas we have immediately that

$$\begin{aligned}\mathbf{T}(z,u\wedge v)&=z_{\mathbf{d}}u^{\mathbf{a}}v^{\mathbf{b}}T_{\mathbf{ab}}^{\mathbf{d}},\\T_{\mathbf{ab}}^{\mathbf{c}}&=\omega_{\mathbf{ab}}^{\mathbf{c}}-\omega_{\mathbf{ba}}^{\mathbf{c}}-c_{\mathbf{ab}}^{\mathbf{c}}\end{aligned} \tag{72}$$

and

$$\begin{aligned}\mathbf{R}(t\wedge z,u\wedge v)&=t^{\mathbf{c}}z_{\mathbf{d}}u^{\mathbf{a}}v^{\mathbf{b}}R_{\mathbf{dab}}^{\mathbf{c}},\\R_{\mathbf{dab}}^{\mathbf{c}}&=e_{\mathbf{a}}(\omega_{\mathbf{bc}}^{\mathbf{d}})-e_{\mathbf{b}}(\omega_{\mathbf{ac}}^{\mathbf{d}})+\omega_{\mathbf{ak}}^{\mathbf{d}}\omega_{\mathbf{bc}}^{\mathbf{k}}-\omega_{\mathbf{bk}}^{\mathbf{d}}\omega_{\mathbf{ac}}^{\mathbf{k}}-c_{\mathbf{ab}}^{\mathbf{k}}\omega_{\mathbf{kc}}^{\mathbf{d}}.\end{aligned} \tag{73}$$

[13]Note that $[\mathbf{u},\mathbf{v}]$ is the standard Lie bracket of vector fields.

Also, we have the very important result. For any $t \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$

$$[\nabla_{e_{\mathbf{a}}}, \nabla_{e_{\mathbf{b}}}]t = \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner t - (T_{\mathbf{ab}}^{\mathbf{c}} - \omega_{\mathbf{ab}}^{\mathbf{c}} + \omega_{\mathbf{ba}}^{\mathbf{c}}) \nabla_{e_{\mathbf{c}}} t. \tag{74}$$

Indeed, using the previous results, we can write

$$\begin{aligned}
[\nabla_{e_{\mathbf{a}}}, \nabla_{e_{\mathbf{b}}}]t &= \frac{1}{2}[\mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}), t] + \nabla_{[e_{\mathbf{a}}, e_{\mathbf{b}}]} t \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner t + \nabla_{[e_{\mathbf{a}}, e_{\mathbf{b}}] - \nabla_{e_{\mathbf{a}}} e_{\mathbf{b}} + \nabla_{e_{\mathbf{b}}} e_{\mathbf{a}}} t + \nabla_{\nabla_{e_{\mathbf{a}}} e_{\mathbf{b}}} t - \nabla_{\nabla_{e_{\mathbf{b}}} e_{\mathbf{a}}} t \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner t + \nabla_{-T_{\mathbf{ab}}^{\mathbf{c}} e_{\mathbf{c}}} t + \nabla_{\omega_{\mathbf{ab}}^{\mathbf{c}} e_{\mathbf{c}}} t - \nabla_{\omega_{\mathbf{ab}}^{\mathbf{c}} e_{\mathbf{c}}} t \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner t - T_{\mathbf{ab}}^{\mathbf{c}} \nabla_{e_{\mathbf{c}}} t + (\omega_{\mathbf{ab}}^{\mathbf{c}} - \omega_{\mathbf{ab}}^{\mathbf{c}}) \nabla_{e_{\mathbf{c}}} t.
\end{aligned} \tag{75}$$

4 Covariant Derivative of Spinor Fields

The spinor bundles introduced in Sections 2.2, 2.3 and 2.4, like $I(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\ell I$, $I = \mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$, and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ (and subbundles) are vector bundles. Thus, as in the case of Clifford fields we can use the general theory of covariant derivative operators on associate vector bundles (described in the Appendices) to obtain formulas for the covariant derivatives of sections of these bundles. Given $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$, we denote the corresponding covariant derivatives by $\nabla^s_V \Psi$ and $\nabla^s_V \Phi$ [14]. We have that for any $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$

$$\nabla^s_V \Psi = \partial_V(\Psi) + \frac{1}{2}\omega_V \Psi, \tag{76}$$

$$\nabla^s_V \Phi = \partial_V(\Phi) - \frac{1}{2}\Phi \omega_V. \tag{77}$$

The proof is analogous to the case of the covaraint derivative of Clifford fields, with the difference that Eq.(48) should be substituted by

$$\Psi^0_{||t} = [(\Xi(0), g(t)a(t))] \text{ and } \Phi^0_{||t} = [(\Xi(0), a(t)g(t)^{-1})].$$

We observe that in the case of (covariant) spinor fields, the matrix representation of the $\omega_{e_{\mathbf{i}}}$ are called *Fock-Ivanenko coefficients.*

Another important result (see [14]) is the following:

Let ∇ be the connection on $\mathcal{C}\ell(M,g)$ to which ∇^s is related. Then, for any $V \in \sec TM$, $A \in \sec \mathcal{C}\ell(M,g)$, $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$,

$$\nabla^s_V(A\Psi) = A(\nabla^s_V \Psi) + (\nabla_V A)\Psi, \tag{78}$$

$$\nabla^s_V(\Phi A) = \Phi(\nabla_V A) + (\nabla^s_V \Phi)A. \tag{79}$$

[14]Recall that $I^l(M) \hookrightarrow C\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ and $I^r(M) \hookrightarrow C\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$.

Using the above results we can prove that the right unit sections associated with spin coframes are *not* constant in any covariant way. Indeed, if $\mathbf{1}^r_\Xi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ is the right unit section associated to the spin coframe Ξ, then

$$\nabla^s_{e_\mathbf{a}} \mathbf{1}^r_\Xi = -\frac{1}{2}\mathbf{1}^r_\Xi \omega_{\mathbf{e_a}}. \tag{80}$$

We now calculate the covariant derivative of a spinor field $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ in the direction of the vector field $V \in \sec TM$ to confirm the validity of Eq.(54).[15]

Let $u : M \to \mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}_{1,3}$ be such that in the spin gauge $\Xi \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ we have for $U \in \sec \mathcal{C}\ell^{(0)}(M,g)$, $UU^{-1} = 1$, $U(x) = [(\Xi(x), u(x))]$. Then, we can write the covariant derivative of $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ in two different gauges $\Xi, \Xi' \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ as

$$\nabla^s_V \Psi = \partial_V(\Psi) + \frac{1}{2}\omega_V \Psi, \tag{81}$$

$$\nabla^s_V \Psi = \partial'_V(\Psi) + \frac{1}{2}\omega'_V \Psi. \tag{82}$$

Now,

$$\Psi = \Psi_I s^I = \Psi'_I s'^I,$$

where s^I, s'^I are the following equivalence classes in $\sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$

$$s^I = \left[\left(\Xi(x), S^I\right)\right]_\ell, \; s'^I = \left[\left(\Xi'(x), S^I\right)\right]_\ell,$$

with S^I a spinor basis in a minimal left ideal in $\mathbb{R}_{1,3}$. Now, if $\Xi' = \Xi u$ we can write, using the fact that $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ is a module over $\mathcal{C}\ell(M,g)$ that

$$\begin{aligned} s'^I &= \left[\left(\Xi'(x), S^I\right)\right]_\ell = \left[\left(\Xi(x)u(x), S^I\right)\right]_\ell = \left[\left(\Xi(x), uS^I\right)\right]_\ell \\ &= \left[(\Xi(x), u(x))\right]_{\mathcal{C}\ell} \left[\left(\Xi(x), S^I\right)\right]_\ell \end{aligned}$$

Now, recalling that $U = [(\Xi(x), u(x))]_{\mathcal{C}\ell} \in \sec \mathcal{C}\ell(M,g)$ we can write

$$s'^I = U s^I. \tag{83}$$

Recalling that

$$\begin{aligned} \partial_V(\Psi) &= V(\Psi_I) s^I, \\ \partial'_V(\Psi) &= \partial_V(\Psi'_I) s'^I, \end{aligned} \tag{84}$$

we get

$$\partial'_V(\Psi) = \partial_V(\Psi) + U^{-1}(\partial_V U). \tag{85}$$

Using Eq.(85) in Eq.(81) and Eq.(82) we immediately confirm the validity of Eq.(54). We explore a little bit more some details in the calculation of $\nabla^s_V \Psi$.

[15] Authors are grateful to Dr. R. A. Mosna for discussions on this issue.

Recall that since $\Psi = [(\Xi, \Psi_\Xi)]_\ell = [(\Xi', \Psi'_\Xi)]_\ell$ we can write

$$[(\Xi', \Psi'_\Xi)]_\ell = [(\Xi u, \Psi'_\Xi)]_\ell = [(\Xi, u\Psi'_\Xi)]_\ell \,,$$

i.e.,

$$\Psi'_\Xi = u^{-1}\Psi_\Xi \tag{86}$$

Then,

$$\partial_V(\Psi) = V(\Psi_I)\left[\left(\Xi, S^I\right)\right]_\ell = \left[\left(\Xi, V(\Psi_I)S^I\right)\right]_\ell = [(\Xi, \partial_V(\Psi_\Xi)] \tag{87}$$

$$\partial'_V(\Psi) = V(\Psi'_I)\left[\left(\Xi, S^I\right)\right]_\ell = \left[\left(\Xi, V(\Psi'_I)S^I\right)\right]_\ell = [(\Xi, \partial_V(\Psi'_\Xi)] \tag{88}$$

and

$$\nabla^s_V\Psi = \partial_V(\Psi) + \frac{1}{2}\omega_V\Psi = \left[\left(\Xi, \partial_V(\Psi_\Xi) + \frac{1}{2}w_V\Psi_\Xi\right)\right]_\ell \,, \tag{89}$$

$$\nabla^s_V\Psi = \partial'_V(\Psi) + \frac{1}{2}\omega'_V\Psi = \left[\left(\Xi', \partial_V(\Psi_{\Xi'}) + \frac{1}{2}w'_V\Psi'_\Xi\right)\right]_\ell \,, \tag{90}$$

with

$$\omega_V = [(\Xi, w_V)]_{\mathcal{C}\ell} \;\;, \tag{91}$$

$$\omega'_V = [(\Xi', w'_V)]_{\mathcal{C}\ell} \,. \tag{92}$$

Then we can write using Eq.(90),

$$\begin{aligned}
\nabla^s_V\Psi &= \left[\left(\Xi u, \partial_V(u^{-1}\Psi_\Xi) + \frac{1}{2}w'_V u^{-1}\Psi_\Xi\right)\right]_\ell \\
&= \left[\left(\Xi, u\partial_V(u^{-1}\Psi_\Xi) + \frac{1}{2}uw'_V u^{-1}\Psi_\Xi\right)\right]_\ell \\
&= \left[\left(\Xi, \partial_V\Psi_\Xi + u\partial_V(u^{-1})\Psi_\Xi + \frac{1}{2}uw'_V u^{-1}\Psi_\Xi\right)\right]_\ell
\end{aligned} \tag{93}$$

Comparing Eqs.(89) and (90) we get

$$\frac{1}{2}w'_V = u^{-1}w_V u - \partial_V(u^{-1})u. \tag{94}$$

Next, we must verify that Eqs.(94) and (54) are compatible. To do this, we

use Eq.(92) to write

$$
\begin{aligned}
\frac{1}{2}\omega'_V &= \frac{1}{2}\left[(\Xi', w'_V)\right]_{\mathcal{C}\ell} = U\frac{1}{2}\omega_V U^{-1} + (\nabla_V U)U^{-1} \\
&= \frac{1}{2}\omega_V + (\partial_V U)U^{-1} \\
&= \left[\left(\Xi, \frac{1}{2}w_V + \partial_V(u)u^{-1}\right)\right]_{\mathcal{C}\ell} \\
&= \left[\left(\Xi' u^{-1}, \frac{1}{2}w_V + \partial_V(u)u^{-1}\right)\right]_{\mathcal{C}\ell} \\
&= \left[\left(\Xi', \frac{1}{2}u^{-1}w_V u + u^{-1}\partial_V(u)\right)\right]_{\mathcal{C}\ell}. \qquad (95)
\end{aligned}
$$

Comparing Eqs.(95) and (94), we see that Eqs.(94) and (54) are indeed compatible.

5 Many Faces of the Dirac Equation

As well known [3, 25, 14], a *covariant* Dirac spinor field is a section $\mathbf{\Psi} \in \sec S_c(M,g) = \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbb{C}^4$. Let $(U = M, \Phi), \Phi(\mathbf{\Psi}) = (x, |\Psi(x)\rangle)$ be a *global* trivialization corresponding to a spin coframe Ξ, such that for $\{e_{\mathbf{a}}\} \in \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$,

$$
\begin{aligned}
s(\Xi) = \{\theta^{\mathbf{a}}\},\ \theta^{\mathbf{a}} \in \sec \mathcal{C}\ell(M,g),\ \theta^{\mathbf{a}}(e_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}} \\
\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \theta^{\mathbf{b}}\theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}},\ \mathbf{a},\mathbf{b} = 0,1,2,3. \qquad (96)
\end{aligned}
$$

The usual Dirac equation in a Lorentzian *spacetime* for the spinor field $\mathbf{\Psi}$ — in interaction with an electromagnetic field $A \in \sec \bigwedge^1(M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ — is then

$$
i\gamma^{\mathbf{a}}(\nabla^s_{e_{\mathbf{a}}} + iqA_{\mathbf{a}})|\Psi(x)\rangle - m|\Psi(x)\rangle = 0, \qquad (97)
$$

where $\underline{\gamma}^{\mathbf{a}} \in \mathbb{C}(4)$, $\mathbf{a} = 0,1,2,3$ is a set of *constant* Dirac *matrices* satisfying

$$
\underline{\gamma}^{\mathbf{a}}\ \underline{\gamma}^{\mathbf{b}} + \ \underline{\gamma}^{\mathbf{b}}\ \underline{\gamma}^{\mathbf{a}} = 2\eta^{\mathbf{ab}}. \qquad (98)
$$

Due to the one-to-one correspondence between *ideal* sections of $\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and of $S_c(M,g)$ as explained above, we can *translate* the Dirac Eq.(97) for a covariant spinor field into an equation for a spinor field, which is a section of $\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, and finally write an equivalent equation for a *DHSF* $\psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$. In order to do that we introduce the spin-Dirac operator.

The *spin-Dirac operator* acting on sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ is the first order differential operator [12]

$$
\boldsymbol{\partial}^s = \theta^{\mathbf{a}}\nabla^s_{e_{\mathbf{a}}}, \qquad (99)
$$

where $\{\theta^{\mathbf{a}}\}$ is a basis as defined in Eq.(96).

Remark 2 *It is crucial to keep in mind the distinction between the Dirac operator* $\boldsymbol{\partial}$ *(Eq.(11)) and the spin-Dirac operator* $\boldsymbol{\partial}^s$ *just defined.*

We now write Dirac equation in $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, denoted $DE\mathcal{C}\ell^l$. It is

$$\boldsymbol{\partial}^s\psi\mathbf{E^{21}} - m\psi\mathbf{E^0} - qA\psi = 0 \tag{100}$$

where $\psi \in \sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ is a *DHSF* and the $\mathbf{E^a} \in \mathbb{R}_{1,3}$ are such that $\mathbf{E^aE^b}+\mathbf{E^bE^a} = 2\eta^{\mathbf{ab}}$. Multiplying Eq.(100) on the right by the idempotent $\mathbf{f} = \frac{1}{2}(1+\mathbf{E^0})\frac{1}{2}(1+i\mathbf{E^2E^1}) \in \mathbb{C}\otimes\mathbb{R}_{1,3}$ we get after some simple algebraic manipulations the following equation for the (complex) ideal left spin-Clifford field $\Psi\mathbf{f} = \Psi \in \sec\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$,

$$i\boldsymbol{\partial}^s\Psi - m\Psi - qA\Psi = 0. \tag{101}$$

Now we can easily show, that given any global trivializations $(U=M,\Theta)$ and $(U=M,\Phi)$, of $\mathcal{C}\ell(M,g)$ and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, there exists matrix representations of the $\{\theta^{\mathbf{a}}\}$ that are equal to the Dirac matrices $\underline{\gamma}^{\mathbf{a}}$ (appearing in Eq.(97)). In that way the correspondence between Eqs.(97), (100) and (101) is proved.

Remark 3 *We must emphasize at this point that we call Eq.(100) the* $DE\mathcal{C}\ell^l$. *It looks similar to the Dirac-Hestenes equation (on Minkowski spacetime) discussed, e.g., in [25], but it is indeed very different regarding its mathematical nature. It is an intrinsic equation satisfied by a legitimate spinor field, namely a DHSF* $\psi \in \sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$. *The question naturally arises: May we write an equation with the same mathematical information of Eq.(100) but satisfied by objects living on the Clifford bundle of an arbitrary Lorentzian spacetime, admitting a spin structure? We now show that the answer to that question is yes.*

5.1 The Dirac-Hestenes Equation

We obtained above a Dirac equation, which we called $DEC\ell^l$, describing the motion of spinor fields represented by sections $\boldsymbol{\Psi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ in interaction with an electromagnetic field $A \in \sec\mathcal{C}\ell(M,\mathtt{g})$,

$$\boldsymbol{\partial}^s\boldsymbol{\Psi}\mathbf{E^{21}} - qA\boldsymbol{\Psi} = m\boldsymbol{\Psi}\mathbf{E^0}, \tag{102}$$

where $\boldsymbol{\partial}^s = \theta^{\mathbf{a}}\nabla^s_{e_{\mathbf{a}}}$, $\{\theta^{\mathbf{a}}\}$ is given by Eq.(96), $\nabla^s_{e_{\mathbf{a}}}$ is the natural spinor covariant derivative acting on $\sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\{\mathbf{E^a}\} \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ is such that $\mathbf{E^aE^b}+\mathbf{E^bE^a} = 2\eta^{\mathbf{ab}}$. As we already mentioned, although Eq.(102) is written in a kind of Clifford bundle (i.e., $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$), it does not suffer from the inconsistency of representing spinors as pure differential forms and, in fact, the object $\boldsymbol{\Psi}$ behaves as it should under Lorentz transformations.

Of course, Eq.(102) is nothing more than a *rewriting* of the usual Dirac equation, where the role of the constant gamma matrices is undertaken by the constant elements $\{\mathbf{E^a}\}$ in $\mathbb{R}_{1,3}$ and by the set $\{\theta^{\mathbf{a}}\}$. In this way, Eq.(102) is *not*

a kind of Dirac-Hestenes equation as discussed, e.g., in [25]. It suffices to say that (i) the state of the electron, represented by $\mathbf{\Psi}$, is not a *Clifford field* and (ii) the $\mathbf{E}^a$'s are just *constant* elements of $\mathbb{R}_{1,3}$ and not sections of 1-form fields in $\mathcal{C}\ell(M,g)$. Nevertheless, as shown originally in [14] Eq.(102) does lead to a multiform Dirac equation once we carefully employ the theory of right and left actions on the various Clifford bundles introduced above. It is that multiform equation that we call the *DHE*.

5.1.1 Representatives of *DHSF* on the Clifford Bundle

Let $\{\mathbf{E}^{\mathbf{a}}\}$ be, as before, a fixed orthonormal basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Remember that these objects are fundamental to the Dirac equation (102) in terms of sections $\mathbf{\Psi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$:

$$\boldsymbol{\partial}^s \mathbf{\Psi}\mathbf{E}^{\mathbf{21}} - qA\mathbf{\Psi} = m\mathbf{\Psi}\mathbf{E}^{\mathbf{0}}.$$

Let ${}_s\Xi \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ be a spin frame and Ξ its dual spin coframe on M and define the sections $\mathbf{1}^l_\Xi$, $\mathbf{1}^r_\Xi$ and $\theta_{\mathbf{a}}$, respectively on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ and $\mathcal{C}\ell(M,g)$, as above. Now we can use Eqs.(45) and (46) to write the above equation in terms of sections of $\mathcal{C}\ell(M,g)$:

$$(\boldsymbol{\partial}^s \mathbf{\Psi})\mathbf{1}^r_\Xi\ \varepsilon^{\mathbf{21}}\mathbf{1}^l_\Xi - qA\mathbf{\Psi} = m\mathbf{\Psi}\mathbf{1}^r_\Xi\ \varepsilon^{\mathbf{0}}\mathbf{1}^l_\Xi. \tag{103}$$

Right-multiplying by $\mathbf{1}^r_\Xi$ yields,

$$\varepsilon^{\mathbf{a}}(\nabla^s_{e_{\mathbf{a}}}\mathbf{\Psi})\mathbf{1}^r_\Xi\varepsilon^{\mathbf{21}} - qA\mathbf{\Psi}\mathbf{1}^r_\Xi = m\mathbf{\Psi}\mathbf{1}^r_\Xi\varepsilon^{\mathbf{0}}. \tag{104}$$

It follows from Eq.(79) that

$$\begin{aligned}(\nabla^s_{e_{\mathbf{a}}}\mathbf{\Psi})\mathbf{1}^r_\Xi &= \nabla_{e_{\mathbf{a}}}(\mathbf{\Psi}\mathbf{1}^r_\Xi) - \mathbf{\Psi}\nabla^s_{e_{\mathbf{a}}}(\mathbf{1}^r_\Xi)\\ &= \nabla_{e_{\mathbf{a}}}(\mathbf{\Psi}\mathbf{1}^r_\Xi) + \frac{1}{2}\mathbf{\Psi}\mathbf{1}^r_\Xi\omega_{e_{\mathbf{a}}},\end{aligned} \tag{105}$$

where Eq.(80) was employed in the last step. Therefore

$$\theta^{\mathbf{a}}\left[\nabla_{e_{\mathbf{a}}}(\mathbf{\Psi}\mathbf{1}^r_\Xi) + \frac{1}{2}\mathbf{\Psi}\mathbf{1}^r_\Xi\omega_{e_{\mathbf{a}}}\right]\theta^{\mathbf{21}} - qA(\mathbf{\Psi}\mathbf{1}^r_\Xi) = m(\mathbf{\Psi}\mathbf{1}^r_\Xi)\theta^{\mathbf{0}}. \tag{106}$$

Thus it is natural to define, for each spin coframe Ξ, the Clifford field $\psi_\Xi \in \sec\mathcal{C}\ell(M,g)$ by

$$\psi_\Xi := \mathbf{\Psi}\mathbf{1}^r_\Xi. \tag{107}$$

We then have

$$\theta^{\mathbf{a}}\left[\nabla_{e_{\mathbf{a}}}\psi_\Xi + \frac{1}{2}\psi_\Xi\omega_{e_{\mathbf{a}}}\right]\theta^{\mathbf{21}} - qA\psi_\Xi = m\psi_\Xi\theta^{\mathbf{0}}. \tag{108}$$

A *comment* about the nature of spinors is in order. As we repeatedly said in the previous sections, spinor fields should not be ultimately regarded as fields

of multiforms, for their behavior under Lorentz transformations is not tensorial (they are able to distinguish between 2π and 4π rotations). So, how can the identification above be correct? The answer is that the definition in Eq.(107) is intrinsically spin-coframe dependent. Clearly, this is the price one ought to pay if one wants to make sense of the procedure of representing spinors by differential forms.

Note also that the covariant derivative acting on ψ_Ξ in Eq.(108) is the tensorial covariant derivative ∇_V on $\mathcal{C}\ell(M,g)$, as it should be. However, we see from the expression above that ∇_V acts on ψ_Ξ together with the term $\frac{1}{2}\psi_\Xi\omega_V$. Therefore, it is natural to define an *effective covariant derivative* $\nabla_V^{(s)}$ acting on ψ_Ξ by

$$\nabla_{e_\mathbf{a}}^{(s)}\psi_\Xi := \nabla_{e_\mathbf{a}}\psi_\Xi + \frac{1}{2}\psi_\Xi\omega_{e_\mathbf{a}}. \tag{109}$$

or, recalling Eq.(76),

$$\nabla_{e_\mathbf{a}}^{(s)}\psi_\Xi = \partial_{e_\mathbf{a}}(\psi_\Xi) + \frac{1}{2}\omega_{e_\mathbf{a}}\psi_\Xi, \tag{110}$$

which emulates the spinorial covariant derivative, as it should. We observe moreover that if $\mathcal{C} \in \sec\mathcal{C}\ell(M,g)$ and if $\psi_\Xi \in \sec\mathcal{C}\ell(M,g)$ is a representative of a Dirac-Hestenes spinor field then

$$\nabla_{e_\mathbf{a}}^{(s)}(\mathcal{C}\psi_\Xi) = (\nabla_{e_\mathbf{a}}\mathcal{C})\,\psi_\Xi + \mathcal{C}(\nabla_{e_\mathbf{a}}^{(s)}\psi_\Xi). \tag{111}$$

With this notation, we finally have the Dirac-Hestenes equation for the *representative* Clifford field $\psi_\Xi \in \sec\mathcal{C}\ell(M,\mathrm{g})$ of a *DHSF* $\mathbf{\Psi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$ relative to the spin coframe Ξ on a Lorentzian spacetime:

$$\theta^\mathbf{a}\nabla_{\mathbf{e_a}}^{(s)}\psi_\Xi\theta^{\mathbf{21}} - qA\psi_\Xi = m\psi_\Xi\theta^{\mathbf{0}}. \tag{112}$$

Let us finally show that this formulation recovers the usual transformation properties characteristic of the Hestenes' formalism as described, e.g., in [25]. Consider then, two spin coframes $\Xi, \Xi' \in \sec\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$, with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \mathrm{Spin}^e_{1,3}$. It follows [14] that $\psi_{\Xi'} = \mathbf{\Psi 1}^r_{\Xi'} = \mathbf{\Psi}u^{-1}\mathbf{1}^r_\Xi = \mathbf{\Psi 1}^r_\Xi U^{-1} = \psi_\Xi U^{-1}$. Therefore, the various spin coframe dependent Clifford fields from Eq.(112) transform as

$$\begin{aligned} \theta'_\mathbf{a} &= U\theta_\mathbf{a}U^{-1}, \\ \psi_{\Xi'} &= \psi_\Xi U^{-1}. \end{aligned} \tag{113}$$

These are exactly the transformation rules one expects from fields satisfying the Dirac-Hestenes equation.

5.1.2 Passive Gauge Invariance of the DHE

It is a trivial exercise to show that if

$$\theta'^\mathbf{a}\nabla_{e'_\mathbf{a}}^{(s)}\psi_{\Xi'}\theta'^{\mathbf{21}} - qA\psi_{\Xi'} = m\psi_{\Xi'}\theta'^{\mathbf{0}}$$

then if the connection ω_V transforms as in Eq.(54) then

$$\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi_\Xi\theta^{\mathbf{21}} - qA\psi_\Xi = m\psi_\Xi\theta^{\mathbf{0}}$$

This property will be referred as *passive gauge invariance* of the DHE. It shows that the fact that even if writing of the Dirac-Hestenes is, of course, frame dependent, this fact does not imply in the selection of *any* preferred reference frame.

The concept of *active* gauge invariance under local rotations of the Dirac-Hestenes equation will be studied below.

5.2 $[\nabla^s_{e_{\mathbf{a}}}, \nabla^s_{e_{\mathbf{b}}}]\psi$

Let $\psi \in \sec \mathcal{C}\ell^{(0)}(M,g)$ be a representative of a Dirac-Hestenes spinor field in a spin frame Ξ defining the orthonormal basis $\{e_{\mathbf{a}}\}$ for TM with $\{\theta^{\mathbf{a}}\}$, ($\theta^{\mathbf{a}} \in \sec \bigwedge^1(T^*M) \hookrightarrow \mathcal{C}\ell(M,g)$) the corresponding dual basis and $\{\theta_{\mathbf{a}}\}$ the reciprocal basis of $\{\theta^{\mathbf{a}}\}$. Then, a trivial calculation similar to the one that leads to Eq.(74) shows that

$$[\nabla^s_{e_{\mathbf{a}}}, \nabla^s_{e_{\mathbf{b}}}]\psi = \frac{1}{2}\mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}})\psi - (T^{\mathbf{c}}_{\mathbf{ab}} - \omega^{\mathbf{c}}_{\mathbf{ab}} + \omega^{\mathbf{c}}_{\mathbf{ba}})\nabla_{e_{\mathbf{c}}}\psi, \tag{114}$$

where $\mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}})$ is the same object as in Eq.(74). This shows the unifying power of our formalism.[16]

We are now prepared to discuss the meaning of local active Lorentz invariance of Maxwell and Dirac-Hestenes Equations.

6 Active Local Lorentz Invariance and Maxwell equations

The action functional for the electromagnetic field F generated by a current J in classical electromagnetic theory is (as it is well-known)

$$\mathcal{A} = \int_U F \wedge \star F - A \wedge \star J. \tag{115}$$

Action (115) is invariant under local active Lorentz transformations. This statement is easy to see once we use the Clifford bundle formalism. Indeed, taking into account that[17]

$$F \wedge \star F = (F \cdot F)\tau_{\mathbf{g}}, \tag{116}$$

we see that if we perform an *active* Lorentz transformation

$$F \mapsto \overset{R}{F} = RFR^{-1}, \tag{117}$$

[16] Also, compare that equation, e.g., with Eq.(6.4.54) of Ramond´s book [22], where some terms are missing.

[17] Analogous argument applies to the term $A \wedge \star J$.

where $R \in \sec \mathrm{Spin}^e_{1,3}(M) \hookrightarrow \sec \mathcal{C}\ell^{(0)}(M,g)$, we have

$$F \wedge \star F = \overset{R}{F} \wedge \star \overset{R}{F}. \tag{118}$$

This follows since $\tau_{\mathbf{g}} = \theta^5$ commutes with even sections of the Clifford bundle and $R\theta^5 R^{-1} = \theta^5$.

Is $\overset{R}{F}$ a solution of Maxwell equations? The answer is in general *negative*. Indeed, if

$$dF = 0, \qquad \delta F = -J, \tag{119}$$

in general,

$$d(RFR^{-1}) \neq 0, \qquad \delta(RFR^{-1}) \neq RJR^{-1}. \tag{120}$$

This can be easily seen in the Clifford bundle formalism, since in general,

$$\boldsymbol{\partial}(RFR^{-1}) \neq R(\boldsymbol{\partial} F)R^{-1}, \tag{121}$$

because, in general, $\theta^{\mathbf{a}} R \neq R\theta^{\mathbf{a}}$.

So, in definitive, we showed that although Maxwell *action* is invariant under active local Lorentz transformations, Maxwell *equation* does not inherit that invariance. However, we can *modify* Maxwell equation in such way that we may propose that F and all RFR^{-1} are gauge equivalent. In order for that to be the case, it is necessary that the Dirac operator be $\boldsymbol{\partial}$ only in a *fiducial* gauge. In other gauges, we have (symbolically)

$$\boldsymbol{\partial}\,()\mapsto \overset{R}{\boldsymbol{\partial}} = R\boldsymbol{\partial}\ \ ()\,R^{-1}. \tag{122}$$

In this case, we always have

$$\overset{R}{\boldsymbol{\partial}}\overset{R}{F} = \overset{R}{J}, \tag{123}$$

and we can say that distinct electromagnetic fields are also classified as distinct equivalence classes, where F and $\overset{R}{F}$ represent the same field in different gauges. These different gauges arises from generalized G-connections (Definition 63). Since this problem appears also in an analogous way in the case of the Dirac equation, we discuss in details, only this latter case. As we will see the suggestion that gauge equivalent equations describe the same physical phenomena implies that the different geometries on (M,$\mathbf{g}$) generated by equivalent gauge connections, with different torsion and curvature tensors are indistinguishable. The empirical use of a particular gauge is then a question of *convenience*.

7 The Dirac-Hestenes Equation on a Riemann-Cartan Spacetime

Let $(M, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ be a general Riemann-Cartan spacetime. In this section we investigate how to obtain a generalization of the Dirac-Hestenes equation (introduced above for a Lorentzian spacetime) for a representative $\psi_{\Xi_u} \in \sec \mathcal{C}\ell(M,g)$

of a Dirac-Hestenes spinor field $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$. In order to do that we first introduce a chart (φ, U) from the maximal atlas of M, with coordinate functions $\{x^\mu\}$. The associate coordinate basis of TU is denoted by $\{e_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu\}$ and we denote by $\{\gamma^\mu = dx^\mu\}$ its dual basis. Moreover, we suppose that $\gamma^\mu \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$. Also, let $\{e_{\mathbf{a}}\} \in \sec \mathrm{P}_{\mathrm{SO}^e_{1,3}}(M)$ an orthonormal frame and $\{\theta^{\mathbf{a}}\}$, $\theta^{\mathbf{a}} \in \sec \bigwedge^1(T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ the corresponding dual basis. In what follows we are going to work in a *fixed* spin coframe $\Xi_u = (u, \{\theta^{\mathbf{a}}\})$ and so, in order to simplify the notation we write simply ψ instead of ψ_{Ξ_u}.

Recall now that the Lagrangian density for a free Dirac-Hestenes spinor in Minkowski spacetime $(M, \eta, \mathring{D}, \tau_\eta, \uparrow)$ [15] can be written in a global Lorentz coordinate chart $\{\mathring{x}^\mu\}$ as

$$\begin{aligned}\mathring{\mathcal{L}}(\mathring{x}, \psi, \mathring{\partial}\psi) &= \mathring{\mathfrak{L}}(\mathring{x}, \psi, \mathring{\partial}\psi) d\mathring{x}^0 \wedge d\mathring{x}^1 \wedge d\mathring{x}^2 \wedge d\mathring{x}^3 \\ &= \left[(\mathring{\partial}\psi \mathbf{i}\gamma_3) \cdot \psi - m\psi \cdot \psi\right] d\mathring{x}^0 \wedge d\mathring{x}^1 \wedge d\mathring{x}^2 \wedge d\mathring{x}^3, \end{aligned} \tag{124}$$

where $\mathring{\partial} = d\mathring{x}^\mu \mathring{D}_{\frac{\partial}{\partial \mathring{x}^\mu}}$. The usual prescription of *minimal coupling* between a given spinor field and the (generalized) gravitational field,

$$d\mathring{x}^\mu \mapsto \theta^{\mathbf{a}},\ \mathring{D}_{\frac{\partial}{\partial \mathring{x}^\mu}} \mapsto \nabla_{e_{\mathbf{a}}},$$

suggests that we take the following Lagrangian for a (representative) ψ of a Dirac-Hestenes spinor field in a Riemann-Cartan spacetime,

$$\begin{aligned}\mathcal{L}(x, \psi, \boldsymbol{\partial}^{(s)}\psi) &= \mathfrak{L}(x, \psi, \boldsymbol{\partial}^{s}\psi) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= \left[(\boldsymbol{\partial}^{(s)}\psi \theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}}) \cdot \psi - m\psi \cdot \psi\right] \sqrt{|\det \mathbf{g}|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{125}$$

where

$$\boldsymbol{\partial}^{(s)}\psi = \theta^{\mathbf{a}} \nabla^{(s)}_{e_{\mathbf{a}}}\psi = \theta^{\mathbf{a}} \left(\partial_{e_{\mathbf{a}}}\psi + \frac{1}{2}\omega_{e_a}\psi\right) \tag{126}$$

Now, in order to use (almost directly) the Lagrangian formalism in *Minkowski* spacetime as presented in [15] for the present problem we must recall that the term $\boldsymbol{\partial}_x \partial_{\boldsymbol{\partial}_x \psi} \mathring{\mathfrak{L}}$ appearing in the Euler-Lagrange equation *(ELE)* for the Dirac-Hestenes spinor field corresponds to $\frac{\partial}{\partial \mathring{x}^\mu}\left(\partial_{\frac{\partial \psi}{\partial \mathring{x}^\mu}} \mathring{\mathfrak{L}}\right)$ as can be proved by direct calculation. If we take this observation into account we immediately obtain that variation of the Lagrangian given by Eq.(125) results in the following *ELE*,

$$\partial_\psi \mathfrak{L} - \partial_\mu \left(\partial_{\partial_\mu \psi} \mathfrak{L}\right) = 0. \tag{127}$$

But, since our Lagrangian is expressed in terms of the Pfaff derivatives $\partial_{e_{\mathbf{a}}}\psi$ we must express $\partial_\mu \left(\partial_{\partial_\mu \psi} \mathfrak{L}\right)$ in terms of them. To do that we first write

$$\theta^{\mathbf{a}} = h^{\mathbf{a}}_\mu dx^\mu. \tag{128}$$

Then, since

$$\sqrt{|\det \mathbf{g}|} = \frac{1}{\det(h_\mu^{\mathbf{a}})} \equiv \mathfrak{h}^{-1} \tag{129}$$

we have

$$\mathfrak{L}(x, \psi, \boldsymbol{\partial}^s \psi) = \left[(\boldsymbol{\partial}^s \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}}) \cdot \psi - m\psi \cdot \psi\right] \mathfrak{h}^{-1}. \tag{130}$$

Also,

$$\partial_{\partial_\mu \psi} \mathfrak{L} = (\partial_{\partial_\mu \psi} \partial_{e_{\mathbf{a}}} \psi)(\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L}) = h_{\mathbf{a}}^\mu \partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L} \tag{131}$$

and Eq.(127) becomes

$$\partial_\psi \mathfrak{L} - (\partial_\mu h_{\mathbf{a}}^\mu) \partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L} - \partial_{e_{\mathbf{a}}} \left(\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L}\right) = 0. \tag{132}$$

Now, we can verify that

$$\mathfrak{h}^{-1} \partial_\mu \mathfrak{h} = h_\sigma^{\mathbf{a}} \partial_\mu h_{\mathbf{a}}^\sigma, \tag{133}$$

and taking into account that

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] = (h_{\mathbf{a}}^\nu \partial_{\mathbf{b}} h_\nu^{\mathbf{c}} - h_{\mathbf{b}}^\nu \partial_{\mathbf{a}} h_\nu^{\mathbf{c}}) e_{\mathbf{c}} = c_{\mathbf{ab}}^{\mathbf{c}} e_{\mathbf{c}} \tag{134}$$

we can write

$$\partial_\mu h_{\mathbf{a}}^\mu = -c_{\mathbf{ab}}^{\mathbf{c}} + \mathfrak{h}^{-1} \partial_{e_{\mathbf{a}}} \mathfrak{h}, \tag{135}$$

and Eq.(132) becomes

$$\partial_\psi \mathfrak{L} - [\partial_{e_{\mathbf{a}}} - c_{\mathbf{ab}}^{\mathbf{c}} + \mathfrak{h}^{-1} \partial_{e_{\mathbf{a}}} \mathfrak{h}] \left(\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L}\right) = 0. \tag{136}$$

Now[18],

$$\partial_\psi \left(\theta^{\mathbf{a}} \omega_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} \cdot \psi\right) = \langle \theta^{\mathbf{a}} \omega_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} + \omega_{e_{\mathbf{a}}} \theta^{\mathbf{a}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} \rangle_\psi,$$

and then

$$\begin{aligned}
\mathfrak{h} \partial_\psi \mathfrak{L} &= \left(\theta^{\mathbf{a}} \partial_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} + \frac{1}{2} \theta^{\mathbf{a}} \omega_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} + \frac{1}{2} \omega_{e_{\mathbf{a}}} \theta^{\mathbf{a}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} - 2m\psi\right), \\
\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L} &= -\mathfrak{h}^{-1} \left(\theta^{\mathbf{a}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}}\right), \\
\partial_{e_{\mathbf{a}}} \left(\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L}\right) &= -\mathfrak{h}^{-1} \partial_{e_{\mathbf{a}}} \mathfrak{h} (\partial_{\partial_{e_{\mathbf{a}}} \psi} \mathfrak{L}) - \mathfrak{h}^{-1} \theta^{\mathbf{a}} \partial_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}},
\end{aligned} \tag{137}$$

and Eq.(136) becomes

$$\theta^{\mathbf{a}} \nabla_{e_{\mathbf{a}}}^{(s)} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} - \frac{1}{4} \theta^{\mathbf{a}} \omega_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} + \frac{1}{4} \theta^{\mathbf{a}} \omega_{e_{\mathbf{a}}} \psi \theta^{\mathbf{0}} \theta^{\mathbf{2}} \theta^{\mathbf{1}} + \frac{1}{2} \theta^{\mathbf{a}} c_{\mathbf{ab}}^{\mathbf{b}} \psi - m\psi = 0 \tag{138}$$

or, recalling that the components of the torsion tensor in an orthonormal basis are given by

$$T_{\mathbf{ab}}^{\mathbf{c}} = \omega_{\mathbf{ab}}^{\mathbf{c}} - \omega_{\mathbf{ba}}^{\mathbf{c}} - c_{\mathbf{ab}}^{\mathbf{c}}, \tag{139}$$

[18] The symbol $\langle\ \rangle_\psi$ means projection on the grades of ψ.

and that, in particular $\omega^{\mathbf{b}}_{\mathbf{ab}} = \eta_{\mathbf{bb}}\omega^{\mathbf{bb}}_{\mathbf{a}} \;\; = 0$, we have

$$
\begin{aligned}
&\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{4}\omega_{e_{\mathbf{a}}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}}_{\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi \\
&= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{4}\theta^{\mathbf{a}}\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{4}\omega_{e_{\mathbf{a}}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}}_{\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi \\
&= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}\theta^{\mathbf{a}}\lrcorner\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}}_{\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi \qquad (140) \\
&= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}T^{\mathbf{b}}_{\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi = 0
\end{aligned}
$$

Finally we write the Dirac-Hestenes equation in a general Riemann-Cartan spacetime as

$$\boldsymbol{\partial}^{(s)}\psi\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}T\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi\theta^{\mathbf{0}} = 0, \qquad (141)$$

where

$$T = T^{\mathbf{b}}_{\mathbf{ab}}\theta^{\mathbf{a}}. \qquad (142)$$

is called the *torsion covector*. Note that in a Lorentzian manifold $T = 0$ and we come back to the Dirac-Hestenes equation given by Eq.(108). We observe moreover that the matrix representation of Eq.(141) coincides with an equation first proposed by Hehl and Datta [9].

We observe yet that if we tried to get the equation of motion of a Dirac-Hestenes spinor field in a Riemann-Cartan spacetime using the principle of minimal coupling in the Dirac-Hestenes equation in Minkowski spacetime, we would miss the term $\frac{1}{2}T\psi\theta^{\mathbf{2}}\theta^{\mathbf{1}}$ appearing in Eq.(141). This would be very bad indeed, because in a complete theory where the $\{\theta^{\mathbf{a}}\}$ and the $\{\omega_{e_{\mathbf{a}}}\}$ must be dynamic fields it can be shown that spinor fields generate torsion (details in [9]).

8 Active Lorentz Invariance of the Dirac-Hestenes Lagrangian

In the proposed gauge theories of the gravitational field (see, e.g., [13, 22]) it is said that the Lagrangians and the corresponding equations of motion of physical fields must be invariant under arbitrary local Lorentz rotations. In this section we briefly investigate how to mathematically implement such a hypothesis and what is its meaning for the case of a Dirac-Hestenes spinor field on a Riemann-Cartan spacetime. The Lagrangian we shall investigate is the one given by Eq.(130), i.e.,

$$\mathfrak{L}(x,\psi,\boldsymbol{\partial}^{(s)}\psi) = \left[(\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}})\cdot\psi - m\psi\cdot\psi\right]\mathfrak{h}^{-1} \qquad \text{(Dirac-Hestenes)}$$

Observe that the Dirac-Hestenes Lagrangian has been written in a fixed (passive gauge) individualized by a spin coframe Ξ and we already know that

it is invariant under passive gauge transformations $\psi \mapsto \psi U^{-1}$ ($UU^{-1} = 1$, $U \in \sec \mathrm{Spin}^e_{1,3}(M) \hookrightarrow \sec \mathcal{C}\ell(M,g)$ once the 'connection' 2-form ω_V transforms as given in Eq.(54), i.e.,

$$\frac{1}{2}\omega_V \mapsto U\frac{1}{2}\omega_V U^{-1} + (\nabla_V U)U^{-1}. \tag{143}$$

Under an active rotation (gauge) transformation the fields transform in new fields given by

$$\begin{aligned}
\psi &\mapsto \psi' = U\psi, \\
\theta^{\mathbf{m}} &\mapsto \theta'^{\mathbf{m}} = U\theta^{\mathbf{m}}U^{-1} = \Lambda^{\mathbf{m}}_{\mathbf{n}}\theta^{\mathbf{n}}, \\
e_{\mathbf{m}} &\mapsto e'_{\mathbf{m}} = (\Lambda^{-1})^{\mathbf{n}}_{\mathbf{m}} e_{\mathbf{n}}.
\end{aligned} \tag{144}$$

Now, according to the ideas of gauge theories, we must search for a new connection ∇'^s such that the Lagrangian results invariant. This will be the case if connections ∇^s and ∇'^s are generalized G-connections (Definition 63), i.e.,

$$\begin{aligned}
\nabla'^{(s)}_{e'_{\mathbf{m}}}(U\psi) &= U\nabla^{(s)}_{e_{\mathbf{m}}}\psi, \\
&\text{or} \\
\nabla'^{(s)}_{e_{\mathbf{n}}}(U\psi) &= \Lambda^{\mathbf{m}}_{\mathbf{n}} U\nabla^{(s)}_{e_{\mathbf{m}}}\psi.
\end{aligned} \tag{145}$$

Also, taking into account the structure of a representative of a spinor covariant derivative in the Clifford bundle (see Eq.(76)) we must have we must have for the Pfaff derivative

$$\partial_{\mathbf{e}_n} \mapsto \partial'_{\mathbf{e}_n} = \Lambda^{\mathbf{m}}_{\mathbf{n}}\partial_{\mathbf{e}_m}, \tag{146}$$

and for the connection,

$$\begin{aligned}
\omega'_{e_{\mathbf{n}}} &= \Lambda^{\mathbf{m}}_{\mathbf{n}}\left(U\omega_{e_{\mathbf{m}}}U^{-1} - 2\partial_{e_{\mathbf{m}}}(U)U^{-1}\right), \\
&\text{or} \\
\omega'_{e'_{\mathbf{m}}} &= U\omega_{e_{\mathbf{m}}}U^{-1} - 2\partial_{e_{\mathbf{m}}}(U)U^{-1}.
\end{aligned} \tag{147}$$

Write

$$\begin{aligned}
\omega'_{e_{\mathbf{n}}} &= \frac{1}{2}\omega'^{\mathbf{kl}}_{\mathbf{m}}\theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}} = \frac{1}{2}\omega'^{\mathbf{kl}}_{\mathbf{m}}\theta_{\mathbf{kl}} \in \sec \mathcal{C}\ell(M,g), \\
\omega_{e_{\mathbf{n}}} &= \frac{1}{2}\omega^{\mathbf{kl}}_{\mathbf{m}}\theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}} = \frac{1}{2}\omega^{\mathbf{kl}}_{\mathbf{m}}\theta_{\mathbf{kl}} \in \sec \mathcal{C}\ell(M,g), \\
U &= \exp F,\ F = \frac{1}{2}F^{\mathbf{rs}}\theta_{\mathbf{rs}} \in \sec \mathcal{C}\ell(M,g).
\end{aligned} \tag{148}$$

Recall that

$$\begin{aligned}
\omega^{\mathbf{rs}}_{\mathbf{n}} &= \eta^{\mathbf{ra}}\omega_{\mathbf{anb}}\eta^{\mathbf{sb}} = \omega^{\mathbf{r}}_{\mathbf{nb}}\eta^{\mathbf{sb}}, \\
\omega^{\mathbf{r}}_{\mathbf{nk}} &= \omega^{\mathbf{rs}}_{\mathbf{n}}\eta_{\mathbf{sk}}.
\end{aligned} \tag{149}$$

Then, from Eqs.(147), (148) and (149) we get

$$\omega'^{\mathbf{r}}_{\mathbf{nk}} = \Lambda^{\mathbf{b}}_{\mathbf{q}} \omega^{\mathbf{p}}_{\mathbf{mb}} \Lambda^{\mathbf{r}}_{\mathbf{p}} \Lambda^{\mathbf{m}}_{\mathbf{k}} - \eta_{\mathbf{sk}} \Lambda^{\mathbf{m}}_{\mathbf{k}} \partial_{e_{\mathbf{m}}} (F^{\mathbf{rs}}) \tag{150}$$

Now, we recall that the components of the torsion tensors $\mathbf{T}$ and $\mathbf{T}'$ of the connections ∇ and ∇' in the orthonormal basis $\{\mathbf{e_r} \otimes \theta^{\mathbf{n}} \wedge \theta^{\mathbf{k}}\}$ are given by

$$\begin{aligned} T^{\mathbf{r}}_{\mathbf{nk}} &= \omega^{\mathbf{r}}_{\mathbf{nk}} - \omega^{\mathbf{r}}_{\mathbf{kn}} - c^{\mathbf{r}}_{\mathbf{nk}}, \\ T'^{\mathbf{r}}_{\mathbf{nk}} &= \omega'^{\mathbf{r}}_{\mathbf{nk}} - \omega'^{\mathbf{r}}_{\mathbf{kn}} - c^{\mathbf{r}}_{\mathbf{nk}}, \end{aligned} \tag{151}$$

where $[e_{\mathbf{n}}, e_{\mathbf{k}}] = c^{\mathbf{r}}_{\mathbf{nk}} e_{\mathbf{r}}$.

Let us suppose that we start with a torsion free connection ∇. This means that $c^{\mathbf{r}}_{\mathbf{nk}} = \omega^{\mathbf{r}}_{\mathbf{nk}} - \omega^{\mathbf{r}}_{\mathbf{kn}}$. Then,

$$T'^{\mathbf{r}}_{\mathbf{nk}} = \Lambda^{\mathbf{b}}_{\mathbf{n}} \Lambda^{\mathbf{m}}_{\mathbf{k}} \Lambda^{\mathbf{r}}_{\mathbf{p}} c^{\mathbf{p}}_{\mathbf{mb}} - c^{\mathbf{r}}_{\mathbf{nk}} - \partial_{e_{\mathbf{m}}} (F^{\mathbf{rs}}) \left[\eta_{\mathbf{sk}} \Lambda^{\mathbf{m}}_{\mathbf{n}} - \eta_{\mathbf{sn}} \Lambda^{\mathbf{m}}_{\mathbf{k}}\right], \tag{152}$$

and we see that $\mathbf{T}' = 0$ only for very particular gauge transformations.

We then arrive at the conclusion that to suppose the Dirac-Hestenes Lagragian is invariant under active rotational gauge transformations implies in an equivalence between torsion free and non torsion free connections. It is always emphasized that in a theory where besides ψ, also the the tetrad fields $\theta^{\mathbf{a}}$ and the connection ω are dynamic variables, the torsion is not zero, because its source is the spin of the ψ field. Well, this is true in particular gauges, because as showed above it seems that it is always possible to find gauges where the torsion is null. Analogous conclusions are valid for the curvature tensors of the 'gauge equivalent connections', as the reader may verify.

A Principal Bundles and Associated Vector Bundles

We recall in this and the next Appendices some of the main definitions and concepts of the theory of principal bundles and their associated vector bundles, including the theory of connections and generalized G-connections in principal and vector bundles, which we shall need in order to introduce the Clifford and spin-Clifford bundles over a Lorentzian manifold. Propositions are in general presented without proofs, which can be found, e.g., in [3, 11, 21].

Definition 4 *A fiber bundle over M with Lie group G is denoted by (E, M, π, G, F). E is a topological space called the total space of the bundle, $\pi : E \to M$ is a continuous surjective map, called the canonical projection and F is the typical fiber. The following conditions must be satisfied:*

a) $\pi^{-1}(x)$, the fiber over x, is homeomorphic to F.

b) Let $\{U_i,\, i \in \mathfrak{I}\}$, where $\mathfrak{I}$ is an index set, be a covering of M, such that:

- *Locally a fiber bundle E is trivial, i.e., it is diffeomorphic to a product bundle, i.e., $\pi^{-1}(U_i) \simeq U_i \times F$ for all $i \in \mathfrak{I}$.*

- *The diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \to U_i \times F$ have the form*

$$\Phi_i(p) = (\pi(p), \phi_{i,x}(p)) \tag{153}$$

$$\phi_i|_{\pi^{-1}(x)} \equiv \phi_{i,x} : \pi^{-1}(x) \to F \text{ is onto} \tag{154}$$

  The collection $\{(U_i, \Phi_i)\}$, $i \in \Im$, are said to be a family of local trivializations for E.

- *The group G acts on the typical fiber. Let $x \in U_i \cap U_j$. Then,*

$$\phi_{j,x} \circ \phi_{i,x}^{-1} : F \to F \tag{155}$$

  must coincide with the action of an element of G for all $x \in U_i \cap U_j$ and $i, j \in \Im$.

- *We call transition functions of the bundle the continuous induced mappings*

$$g_{ij} : U_i \cap U_j \to G, \text{ where } g_{ij}(x) = \phi_{i,x} \circ \phi_{j,x}^{-1}. \tag{156}$$

For consistence of the theory the transition functions must satisfy the cocycle condition

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x). \tag{157}$$

Definition 5 *$(P, M, \pi, G, F \equiv G) \equiv (P, M, \pi, G)$ is called a principal fiber bundle* (PFB) *if all conditions in 4 are fulfilled and moreover, if there is a right action of G on elements $p \in P$, such that:*

a) the mapping (defining the right action) $P \times G \ni (p, g) \mapsto pg \in P$ is continuous.

b) given $g, g' \in G$ and $\forall p \in P$, $(pg)g' = p(gg')$.

c) $\forall x \in M, \pi^{-1}(x)$ is invariant under the action of G, i.e., each element of $p \in \pi^{-1}(x)$ is mapped into $pg \in \pi^{-1}(x)$, i.e., it is mapped into an element of the same fiber.

d) G acts free and transitively on each fiber $\pi^{-1}(x)$, which means that all elements within $\pi^{-1}(x)$ are obtained by the action of all the elements of G on any given element of the fiber $\pi^{-1}(x)$. This condition is, of course, necessary for the identification of the typical fiber with G.

Definition 6 *A bundle $(E, M, \pi_1, G = Gl(m, \mathcal{F}), F = \mathbf{V})$, where $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$ (respectively the real and complex fields), $Gl(m, \mathcal{F})$ is the linear group, and $\mathbf{V}$ is an m-dimensional vector space over $\mathcal{F}$ is called a vector bundle.*

Definition 7 *A vector bundle (E, M, π, G, F) denoted $E = P \times_\rho F$ is said to be associated to a* PFB *bundle (P, M, π, G) by the linear representation ρ of G in $F = \mathbf{V}$ (a linear space of finite dimension over an appropriate field, which is called the* carrier space *of the representation) if its transition functions are the images under ρ of the corresponding transition functions of the* PFB

(P, M, π, G). *This means the following: consider the following local trivializations of P and E respectively*

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times G, \tag{158}$$

$$\Xi_i : \pi_1^{-1}(U_i) \to U_i \times F, \tag{159}$$

$$\Xi_i(q) = (\pi_1(q), \chi_i(q)) = (x, \chi_i(q)), \tag{160}$$

$$\chi_i|_{\pi_1^{-1}(x)} \equiv \chi_{i,x} : \pi_1^{-1}(x) \to F, \tag{161}$$

where $\pi_1 : P \times_\rho F \to M$ is the projection of the bundle associated to (P, M, π, G). Then, for all $x \in U_i \cap U_j$, $i, j \in \mathfrak{I}$, we have

$$\chi_{j,x} \circ \chi_{i,x}^{-1} = \rho(\phi_{j,x} \circ \phi_{i,x}^{-1}). \tag{162}$$

In addition, the fibers $\pi^{-1}(x)$ are vector spaces isomorphic to the representation space V.

Definition 8 *Let (E, M, π, G, F) be a fiber bundle and $U \subset M$ an open set. A local section of the fiber bundle (E, M, π, G, F) on U is a mapping*

$$s : U \to E \quad \text{such that} \quad \pi \circ s = Id_U, \tag{163}$$

If $U = M$ we say that s is a global section.

Remark 9 *There is a relation between sections and local trivializations for principal bundles. Indeed, each local section s, (on $U_i \subset M$) for a principal bundle (P, M, π, G) determines a local trivialization $\Phi_i : \pi^{-1}(U) \to U \times G$, of P by setting*

$$\Phi_i^{-1}(x, g) = s(x)g = pg = R_g p. \tag{164}$$

Conversely, Φ_i determines s since

$$s(x) = \Phi_i^{-1}(x, e). \tag{165}$$

Proposition 10 *A principal bundle is trivial if and only if it has a global cross section.*

Proposition 11 *A vector bundle is trivial if and only if its associated principal bundle is trivial.*

Proposition 12 *Any fiber bundle (E, M, π, G, F) such that M is a paracompact manifold and the fiber F is a vector space admits a cross section.*

Remark 13 *Then, any vector bundle associated to a trivial principal bundle has non-zero global sections. Note however that a vector bundle may admit a non-zero global section even if it is not trivial. Indeed, as shown in the main text, any Clifford bundle possesses a global identity section, and some spin-Clifford bundles admits also identity sections once a trivialization is given.*

Definition 14 *We say that the structure group G' of a fiber bundle (E, M, π, G, F) is reducible to G if the bundle admits an equivalent structure defined with a subgroup G' of the structure group G. More precisely, this means that the fiber bundle admits a family of local trivializations such that the transition functions takes values in G', i.e., $g_{ij} : U_i \cap U_j \to G'$.*

A.1 Frame Bundle

The tangent bundle TM to a differentiable n-dimensional manifold M is an associated bundle to a principal bundle called the frame bundle $F(M) = \bigcup_{x \in M} F_x M$, where $F_x M$ is the set of frames at $x \in M$. Let $\{x^i\}$ be the coordinate functions associated to a local chart (U_i, φ_i) of the maximal atlas of M. Then, $T_x M$ has a natural basis $\{ \frac{\partial}{\partial x^i}\big|_x \}$ on $U_i \subset M$.

Definition 15 *A frame at $T_x M$ is a set $\Sigma_x = \{ e_1|_x , ..., e_n|_x \}$ of linearly independent vectors such that*

$$e_i|_x = F_i^j \left. \frac{\partial}{\partial x^j} \right|_x , \tag{166}$$

and where the matrix (F_i^j) with entries $A_i^j \in \mathbb{R}$, belongs to the the real general linear group in n dimensions $Gl(n, \mathbb{R})$. We write $(F_i^j) \in Gl(n, \mathbb{R})$.

A local trivialization $\phi_i : \pi^{-1}(U_i) \to U_i \times Gl(n, \mathbb{R})$ is defined by

$$\phi_i(f) = (x, \Sigma_x),\ \pi(f) = x. \tag{167}$$

The action of $a = (a_i^j) \in Gl(n, \mathbb{R})$ on a frame $f \in F(U)$ is given by $(f, a) \to fa$, where the new frame $fa \in F(U)$ is defined by $\phi_i(fa) = (x, \Sigma'_x)$, $\pi(f) = x$, and

$$\begin{aligned} \Sigma'_x &= \{ e'_1|_x , ..., e'_n|_x \}, \\ e'_i|_x &= e_j|_x \, a_i^j. \end{aligned} \tag{168}$$

Conversely, given frames Σ_x and Σ'_x there exists $a = (a_i^j) \in Gl(n, \mathbb{R})$ such that Eq.(168) is satisfied, which means that $Gl(n, \mathbb{R})$ acts on $F(M)$ actively.

Let $\{x^i\}$ and $\{\bar{x}^i\}$ be the coordinate functions associated to the local chart (U_i, φ_i) (U_i, φ_i) and of the maximal atlas of M. If $x \in U_i \cap U_j$ we have

$$\begin{aligned} e_i|_x &= F_i^j \left. \frac{\partial}{\partial x^j} \right|_x = \bar{F}_i^j \left. \frac{\partial}{\partial \bar{x}^j} \right|_x , \\ (F_i^j), (\bar{F}_i^j) &\in Gl(n, \mathbb{R}). \end{aligned} \tag{169}$$

Since $F_i^j = \bar{F}_k^j \left(\frac{\partial x^k}{\partial \bar{x}^i} \right) \Big|_x$ we have that the transition functions are

$$g_i^k(x) = \left. \left(\frac{\partial x^k}{\partial \bar{x}^i} \right) \right|_x \in Gl(n, \mathbb{R}). \tag{170}$$

Remark 16 *Given $U \subset M$ we shall denote by $\Sigma \in \sec F(U)$ a section of $F(U)$ $\subset F(M)$. This means that given a local trivialization $\phi : \pi^{-1}(U) \to U \times Gl(n, \mathbb{R})$, $\phi(\Sigma) = (x, \Sigma_x)$, $\pi(f) = x$. Sometimes, we also use the sloppy notation $\{e_i\} \in$ $\sec F(U)$ or even $\{e_i\} \in \sec F(M)$ when the context is clear.*

A.2 Orthonormal Frame Bundle

Suppose that the manifold M is equipped with a metric field $g \in \sec T^{2,0}M$ of signature (p,q), $p+q=n$. Then, we can introduce *orthonormal* frames in each T_xU. In this case we denote an orthonormal frame by $\Sigma_x = \{\mathbf{e}_1|_x, ..., \mathbf{e}_n|_x\}$ and

$$\mathbf{e}_i|_x = h_i^j \left. \frac{\partial}{\partial x^j} \right|_x , \tag{171}$$

$$g(\mathbf{e}_i|_x, \mathbf{e}_j|_x)\big|_x = \mathrm{diag}(1, 1, ..., 1, -1, ..., -1) \tag{172}$$

with $(h_i^j) \in \mathrm{O}(p,q)$, the real orthogonal group in n dimensions. In this case we say that the frame bundle has been reduced to the *orthonormal frame bundle*, which will be denoted by $\mathbf{P}_{\mathrm{O}(n)}(M)$. A section $\Sigma \in \sec \mathbf{P}_{\mathrm{O}(n)}(U)$ is called a vierbein.

Remark 17 *The principal bundle of orthonormal frames $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ over a Lorentzian manifold modelling spacetime and its covering bundle called spin bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ discussed in Section 2 play an important role in this article. Also, vector bundles associated to these bundles are very important. Associated to $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ we have the tensor bundle, the exterior bundle and the Clifford bundle. Associated to $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ we have several spinor bundles, in particular the spin-Clifford bundle, residence of the Dirac-Hestenes spinor fields. All those bundles and their relations are studied in details in [14] and briefly discussed in Section 2.*

Given two vector bundles $(E, M, \pi, G, \mathbf{V})$ and $(E', M', \pi', G', \mathbf{V}')$ we have

Definition 18 *The product bundle $E \times E'$ is a fiber bundle whose basis space is $M \times M'$, the typical fiber is $\mathbf{V} \oplus \mathbf{V}'$, the structural group of $E \times E'$ acts separately as G and G' in each one of the components of $\mathbf{V} \oplus \mathbf{V}'$ and the projection $\pi \times \pi'$ is such that $E \times E' \overset{\pi\times\pi'}{\to} M \times M'$.*

Definition 19 *Given two vector bundles $(E, M, \pi, G, \mathbf{V})$ and $(E', M, \pi', G', \mathbf{V}')$ over the same basis space, the Whitney sum bundle $E \oplus E'$ is the pullback of $E \times E'$ by $h : M \to M \times M$, $h(p) = (p,p)$.*

Definition 20 *Given two vector bundles $(E, M, \pi, G, \mathbf{V})$ and $(E', M, \pi', G', \mathbf{V}')$ over the same basis space, the tensor product bundle $E \otimes E'$ is the bundle obtained from E and E' by assigning the tensor product of fibers $\pi_x^{-1} \otimes \pi_x'^{-1}$ for all $x \in M$.*

Remark 21 *With the above definitions we can easily show that given three vector bundles, say, E, E', E'' we have*

$$E \oplus (E' \otimes E'') = (E \otimes E') \oplus (E \otimes E'') \tag{173}$$

B Equivalent Definitions of a Connection in Principal Bundles

To define the concept of a *connection* on a *PFB* (P, M, π, G), we recall that since $\dim(M) = m$, if $\dim(G) = n$, then $\dim(P) = n + m$. Obviously, for all $x \in M$, $\pi^{-1}(x)$ is an n-dimensional submanifold of P diffeomorphic to the structure group G and π is a submersion, $\pi^{-1}(x)$ is a closed submanifold of P for all $x \in M$.

The tangent space T_pP, $p \in \pi^{-1}(x)$, is an $(n+m)$-dimensional vector space and the tangent space $V_pP \equiv T_p(\pi^{-1}(x))$ to the fiber over x at the same point $p \in \pi^{-1}(x)$ is an n-dimensional linear subspace of T_pP called the *vertical subspace* of T_pP[19].

Now, roughly speaking a connection on P is a rule that makes possible a *correspondence* between any two fibers along a curve $\sigma : \mathbb{R} \supseteq I \to M, t \mapsto \sigma(t)$. If p_0 belongs to the fiber over the point $\sigma(t_0) \in \sigma$, we say that p_0 is parallel translated along σ by means of this *correspondence*.

Definition 22 *A horizontal lift of σ is a curve $\hat{\sigma} : \mathbb{R} \supseteq I \to P$ (described by the parallel transport of p).*

It is intuitive that such a transport takes place in P along directions specified by vectors in T_pP, which do not lie within the vertical space V_pP. Since the tangent vectors to the paths of the basic manifold passing through a given $x \in M$ span the entire tangent space T_xM, the corresponding vectors $Y_p \in T_pP$ (in whose direction parallel transport can generally take place in P) span a four-dimensional linear subspace of T_pP called the *horizontal space* of T_pP and denoted by H_pP. Now, the mathematical concept of a connection can be presented. This is done through three equivalent definitions given below which encode rigorously the intuitive discussion given above. We have,

Definition 23 *A connection on a* PFB (P, M, π, G) *is an assignment to each $p \in P$ of a subspace $H_pP \subset T_pP$, called the horizontal subspace for that connection, such that H_pP depends smoothly on p and the following conditions hold:*

(i) $\pi_* : H_pP \to T_xM$, $x = \pi(p)$, is an isomorphism.
(ii) H_pP depends smoothly on p.
(iii) $(R_g)_* H_pP = H_{pg}P, \forall g \in G, \forall p \in P$.

Here we denote by π_* the *differential* of the mapping π and by $(R_g)_*$ the differential of the mapping $R_g : P \to P$ (the right action) defined by $R_g(p) = pg$.

Since $x = \pi(\hat{\sigma}(t))$ for any curve in P such that $\hat{\sigma}(t) \in \pi^{-1}(x)$ and $\hat{\sigma}(0) = p_0$, we conclude that π_* maps all vertical vectors in the zero vector in T_xM, i.e.,

[19] Here we may be tempted to realize that as it is possible to construct the vertical space for all $p \in P$ then we can define a horizontal space as the complement of this space in respect to T_pP. Unfortunately this is not so, because we need a smoothly association of a horizontal space in every point. This is possible only by means of a connection.

$\pi_*(V_pP) = 0$ and we have[20],

$$T_pP = H_pP \oplus V_pP. \tag{174}$$

Then every $Y_p \in T_pP$ can be written as

$$\mathbf{Y} = \mathbf{Y}_p^h + \mathbf{Y}_p^v, \qquad \mathbf{Y}_p^h \in H_pP, \qquad \mathbf{Y}_p^v \in V_pP. \tag{175}$$

Therefore, given a vector field Y over M it is possible to lift it to a horizontal vector field over P, i.e., $\pi_*(Y_p) = \pi_*(Y_p^h) = Y_x \in T_xM$ for all $p \in P$ with $\pi(p) = x$. In this case, we call Y_p^h *horizontal lift* of Y_x. We say moreover that Y is a horizontal vector field over P if $Y^h = Y$.

Definition 24 *A* connection *on a* PFB (P, M, π, G) *is a mapping* $\Gamma_p : T_xM \to T_pP$*, such that* $\forall p \in P$ *and* $x = \pi(p)$ *the following conditions hold:*

(i) Γ_p is linear.
(ii) $\pi_* \circ \Gamma_p = Id_{T_xM}$.
(iii) the mapping $p \mapsto \Gamma_p$ is differentiable.
(iv) $\Gamma_{R_gp} = (R_g)_*\Gamma_p$, for all $g \in G$.
We need also the concept of parallel transport. It is given by

Definition 25 *Let* $\sigma : \ni I \to M$, $t \mapsto \sigma(t)$ *with* $x_0 = \sigma(0) \in M$*, be a curve in* M *and let* $p_0 \in P$ *such that* $\pi(p_0) = x_0$*. The parallel transport of* p_0 *along* σ *is given by the curve* $\hat{\sigma} : \ni I \to P, t \mapsto \hat{\sigma}(t)$ *defined by*

$$\frac{d}{dt}\hat{\sigma}(t) = \Gamma_p(\frac{d}{dt}\sigma(t)), \tag{176}$$

with $p_0 = \hat{\sigma}(0)$ *and* $\hat{\sigma}(t) = p_{\parallel t}$, $\pi(p_{\parallel t}) = x$.

In order to present yet a *third* definition of a connection we need to know more about the nature of the vertical space V_pP. For this, let $Y \in T_eG = \mathfrak{G}$ be an element of the Lie algebra $\mathfrak{G}$ of G. The vector Y is the tangent to the curve produced by the exponential map

$$\mathbf{Y} = \left.\frac{d}{dt}\left(\exp(t\mathbf{Y})\right)\right|_{t=0}. \tag{177}$$

Then, for every $p \in P$ we can attach to each $Y \in T_eG = \mathfrak{G}$ a unique element $Y_p^v \in V_pP$ as follows: let $f : (-\varepsilon, \varepsilon) \to P$ be a curve in P and $\mathfrak{F} : P \to \mathbb{R}$ a smooth function and consider the function $\mathfrak{F} \circ f(t) = \mathfrak{F}(p \exp tY)$. Then we have

$$\mathbf{Y}_p^v(\mathfrak{F}) = \left.\frac{d}{dt}\mathfrak{F}\left(p\exp(t\mathbf{Y})\right)\right|_{t=0}. \tag{178}$$

[20] We also write $TP = HP \oplus VP$.

By this construction we attach to each $Y \in T_eG = \mathfrak{G}$ a unique vector field over P, called the fundamental field corresponding to this element. We then have the canonical isomorphism

$$\mathbf{Y}_p^v \longleftrightarrow \mathbf{Y}, \quad \mathbf{Y}_p^v \in V_pP, \quad \mathbf{Y} \in T_eG = \mathfrak{G} \tag{179}$$

from which we get

$$V_pP \simeq \mathfrak{G}. \tag{180}$$

Definition 26 *A connection on a* PFB (P, M, π, G) *is a 1-form field* ω *on* P *with values in the Lie algebra* $\mathfrak{G} = T_eG$ *such that* $\forall p \in P$ *we have,*

(i) $\boldsymbol{\omega}_p(Y_p^v) = Y$ and $Y_p^v \longleftrightarrow Y$, where $Y_p^v \in V_pP$ and $Y \in T_eG = \mathfrak{G}$.
(ii) $\boldsymbol{\omega}_p$ depends smoothly on p.
(iii) $\boldsymbol{\omega}_p[(R_g)_*Y_p] = (\mathrm{Ad}_{g^{-1}}\omega_p)(Y_p)$, where $\mathrm{Ad}_{g^{-1}}\omega_p = g^{-1}\omega_p g$.
It follows that if $\{\mathcal{G}_a\}$ is a basis of $\mathfrak{G}$ and $\{\theta^i\}$ is a basis for T^*P then

$$\boldsymbol{\omega}_p = \omega_p^a \otimes \mathcal{G}_a = \omega_i^a(p)\theta_p^i \otimes \mathcal{G}_a, \tag{181}$$

where ω^a are 1-forms on P.

Then the horizontal spaces can be defined by

$$H_pP = \ker(\boldsymbol{\omega}_p), \tag{182}$$

which shows the equivalence between the definitions.

B.1 The Connection on the Base Manifold

Definition 27 *Let* $U \subset M$ *and*

$$s : U \to \pi^{-1}(U) \subset P, \qquad \pi \circ s = Id_U, \tag{183}$$

be a local section of the *PFB* (P, M, π, G).

Definition 28 *Let* $\boldsymbol{\omega}$ *be a connection on* P*. The 1-form* $s^*\boldsymbol{\omega}$ *(the pullback of* $\boldsymbol{\omega}$ *under* s*) by*

$$(s^*\boldsymbol{\omega})_x(Y_x) = \boldsymbol{\omega}_{s(x)}(s_*Y_x), \; Y_x \in T_xM, \; s_*Y_x \in T_pP, \quad p = s(x), \tag{184}$$

is called the local gauge potential.

It is quite clear that $s^*\boldsymbol{\omega} \in \sec T^*U \otimes \mathfrak{G}$. This object differs from the *gauge field* used by physicists by numerical constants (with units). Conversely we have the following

Proposition 29 *Given* $\omega \in \sec T^*U \otimes \mathfrak{G}$ *and a differentiable section of* $\pi^{-1}(U) \subset P$, $U \subset M$, *there exists one and only one connection* $\boldsymbol{\omega}$ *on* $\pi^{-1}(U)$ *such that* $s^*\boldsymbol{\omega} = \omega$.

Consider now

$$\begin{aligned} \omega \in T^*U \otimes \mathfrak{G}, \quad \omega = (\Phi^{-1}(x,e))^* \boldsymbol{\omega} = s^* \boldsymbol{\omega}, \quad s(x) = \Phi^{-1}(x,e), \\ \omega' \in T^*U' \otimes \mathfrak{G}, \quad \omega' = (\Phi'^{-1}(x,e))^* \boldsymbol{\omega} = s'^* \boldsymbol{\omega}, \quad s'(x) = \Phi'^{-1}(x,e). \end{aligned} \tag{185}$$

Then we can write, for each $p \in P$ ($\pi(p) = x$), parameterized by the local trivializations Φ and Φ' respectively as (x,g) and (x,g') with $x \in U \cap U'$, that

$$\omega_p = g^{-1}dg + g^{-1}\omega_x g = g'^{-1}dg' + g'^{-1}\omega'_x g'. \tag{186}$$

Now, if

$$g' = hg, \tag{187}$$

we immediately get from Eq.(186) that

$$\omega'_x = hdh^{-1} + h\omega_x h^{-1}, \tag{188}$$

which can be called the *transformation law* for the gauge fields.

C Exterior Covariant Derivatives.

Let $\overset{k}{\bigwedge}(P,\mathfrak{G}) = \overset{k}{\bigwedge}(T^*P) \otimes \mathfrak{G}, 0 \leq k \leq n$, be the set of all k-form fields over P with values in the Lie algebra $\mathfrak{G}$ of the gauge group G (and, of course, the connection $\boldsymbol{\omega} \in \sec \overset{1}{\bigwedge}(P,\mathfrak{G})$).

Definition 30 *For each* $\boldsymbol{\varphi} \in \sec \overset{k}{\bigwedge}(P,\mathfrak{G})$ *we define the so called* horizontal form $\boldsymbol{\varphi}^h \in \sec \overset{k}{\bigwedge}(P,\mathfrak{G})$ *by*

$$\boldsymbol{\varphi}^h_p(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) = \boldsymbol{\varphi}(\mathbf{X}^h_1, \mathbf{X}^h_2, ..., \mathbf{X}^h_k), \tag{189}$$

where $\mathbf{X}_i \in T_pP$, $i = 1, 2, .., k$.

Notice that $\boldsymbol{\varphi}^h_p(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) = 0$ if one (or more) of the $\mathbf{X}_i \in T_pP$ are vertical.

Definition 31 $\boldsymbol{\varphi} \in \sec \overset{k}{\bigwedge}(T^*P) \otimes \mathbf{V}$ *(where* $\mathbf{V}$ *is a vector space) is said to be horizontal if* $\boldsymbol{\varphi}_p(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) = 0$*, if at least one (say)* $\mathbf{X}_i \in T_pP$ *is vertical.*

Definition 32 $\boldsymbol{\varphi} \in \sec \overset{k}{\bigwedge}(T^*P) \otimes \mathbf{V}$ *is said to be of type* $(\rho, \mathbf{V})$ *if* $\forall g \in G$ *we have*

$$R^*_g \boldsymbol{\varphi} = \rho(g^{-1})\boldsymbol{\varphi}. \tag{190}$$

Definition 33 *Let* $\boldsymbol{\varphi} \in \sec \overset{k}{\bigwedge}(T^*P) \otimes \mathbf{V}$ *be horizontal. Then,* $\boldsymbol{\varphi}$ *is said to be tensorial of type* $(\rho, \mathbf{V})$.

Definition 34 *The exterior covariant derivative of* $\boldsymbol{\varphi} \in \sec \overset{k}{\bigwedge}(P, \mathfrak{G})$ *in relation to the connection* ω *is*

$$D^{\boldsymbol{\omega}}\boldsymbol{\varphi} = (d\boldsymbol{\varphi})^h \in \sec \bigwedge^{k+1}(P, \mathfrak{G}), \tag{191}$$

where $D^{\boldsymbol{\omega}}\boldsymbol{\varphi}_p(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k, \mathbf{X}_{k+1}) = d\boldsymbol{\varphi}_p(\mathbf{X}_1^h, \mathbf{X}_2^h, ..., \mathbf{X}_k^h, \mathbf{X}_{k+1}^h)$. *Notice that* $d\boldsymbol{\varphi} = d\boldsymbol{\varphi}^a \otimes \mathcal{G}_a$ *where* $\boldsymbol{\varphi}^a \in \sec \overset{k}{\bigwedge}(P)$, $a = 1, 2, ..., n$.

Definition 35 *The commutator of* $\boldsymbol{\varphi} \in \sec \overset{i}{\bigwedge}(P, \mathfrak{G})$ *and* $\boldsymbol{\psi} \in \sec \overset{j}{\bigwedge}(P, \mathfrak{G})$, $0 \leq i, j \leq n$ *by* $[\boldsymbol{\varphi}, \boldsymbol{\psi}] \in \sec \overset{i+j}{\bigwedge}(P, \mathfrak{G})$ *such that if* $\mathbf{X}_1, ..., \mathbf{X}_{i+j} \in \sec TP$, *then*

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}](\mathbf{X}_1, ..., \mathbf{X}_{i+j}) = \frac{1}{i!j!} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} [\boldsymbol{\varphi}(\mathbf{X}_{\iota(1)}, ..., \mathbf{X}_{\iota(i)}), \boldsymbol{\psi}(\mathbf{X}_{\iota(i+1)}, ..., \mathbf{X}_{\iota(i+j)})], \tag{192}$$

where $\mathcal{S}_n$ *is the permutation group of* n *elements and* $(-1)^{\sigma} = \pm 1$ *is the sign of the permutation. The brackets* [,] *in the second member of Eq.(192) are the Lie brackets in* $\mathfrak{G}$.

By writing

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\psi} = \boldsymbol{\psi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\varphi}^a \in \sec \bigwedge^i(T^*P), \quad \boldsymbol{\psi}^a \in \sec \bigwedge^j(T^*P), \tag{193}$$

we can write

$$\begin{aligned} [\boldsymbol{\varphi}, \boldsymbol{\psi}] &= \boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b \otimes [\mathcal{G}_a, \mathcal{G}_b] \\ &= f_{ab}^c (\boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b) \otimes \mathcal{G}_c \end{aligned} \tag{194}$$

where f_{ab}^c are the structure constants of the Lie algebra .

With Eq.(194) we can prove *easily* the following important properties involving commutators:

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = (-)^{1+ij}[\boldsymbol{\psi}, \boldsymbol{\varphi}], \tag{195}$$

$$(-1)^{ik}[[\boldsymbol{\varphi}, \boldsymbol{\psi}], \boldsymbol{\tau}] + (-1)^{ji}[[\boldsymbol{\psi}, \boldsymbol{\tau}], \boldsymbol{\varphi}] + (-1)^{kj}[[\boldsymbol{\tau}, \boldsymbol{\varphi}], \boldsymbol{\psi}] = 0, \tag{196}$$

$$d[\boldsymbol{\varphi}, \boldsymbol{\psi}] = [d\boldsymbol{\varphi}, \boldsymbol{\psi}] + (-1)^i[\boldsymbol{\varphi}, d\boldsymbol{\psi}]. \tag{197}$$

for $\boldsymbol{\varphi} \in \sec \overset{i}{\bigwedge}(P, \mathfrak{G})$, $\boldsymbol{\psi} \in \sec \overset{j}{\bigwedge}(P, \mathfrak{G})$, $\boldsymbol{\tau} \in \sec \overset{k}{\bigwedge}(P, \mathfrak{G})$.

We shall also need the following identity

$$[\boldsymbol{\omega},\boldsymbol{\omega}](\mathbf{X_1},\mathbf{X_2}) = 2[\boldsymbol{\omega}(\mathbf{X}_1),\boldsymbol{\omega}(\mathbf{X}_2)]. \tag{198}$$

The proof of Eq.(198) is given as follows:
(i) Recall that

$$[\boldsymbol{\omega},\boldsymbol{\omega}] = (\boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b) \otimes [\mathcal{G}_a,\mathcal{G}_b]. \tag{199}$$

(ii) Let $\mathbf{X}_1, \mathbf{X}_2 \in \sec TP$ (i.e., $\mathbf{X}_1$ and $\mathbf{X}_2$ are vector fields on P). Then,

$$\begin{aligned}[\boldsymbol{\omega},\boldsymbol{\omega}](\mathbf{X_1},\mathbf{X_2}) &= (\boldsymbol{\omega}^a(\mathbf{X}_1)\boldsymbol{\omega}^b(\mathbf{X_2}) - \boldsymbol{\omega}^a(\mathbf{X_2})\boldsymbol{\omega}^b(\mathbf{X_1}))[\mathcal{G}_a,\mathcal{G}_b] \\ &= 2[\boldsymbol{\omega}(\mathbf{X}_1),\boldsymbol{\omega}(\mathbf{X}_2)].\end{aligned} \tag{200}$$

Definition 36 *The curvature form of the connection* $\boldsymbol{\omega} \in \sec \overset{1}{\bigwedge}(P,\mathfrak{G})$ *is* $\boldsymbol{\Omega}^{\boldsymbol{\omega}} \in \sec \overset{2}{\bigwedge}(P,\mathfrak{G})$ *defined by*

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega}. \tag{201}$$

Definition 37 *The connection* $\boldsymbol{\omega}$ *is said to be flat if* $\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0$.

Proposition 38

$$D^{\boldsymbol{\omega}}\boldsymbol{\omega}(\mathbf{X_1},\mathbf{X_2}) = d\boldsymbol{\omega}(\mathbf{X}_1,\mathbf{X}_2) + [\boldsymbol{\omega}(\mathbf{X}_1),\boldsymbol{\omega}(\mathbf{X}_2)]. \tag{202}$$

Eq.(202) can be written using Eq.(200) (and recalling that $\omega(\mathbf{X}) = \omega^a(\mathbf{X})\mathcal{G}_a$). Thus we have

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]. \tag{203}$$

Proposition 39 (Bianchi identity):

$$D\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0. \tag{204}$$

Proof. (i) Let us calculate $d\boldsymbol{\Omega}^{\boldsymbol{\omega}}$. We have,

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = d\left(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]\right). \tag{205}$$

We now take into account that $d^2\boldsymbol{\omega} = 0$ and that from the properties of the commutators given by Eqs.(195), (205) and (197) above, we have

$$\begin{aligned}d[\boldsymbol{\omega},\boldsymbol{\omega}] &= [d\boldsymbol{\omega},\boldsymbol{\omega}] - [\boldsymbol{\omega},d\boldsymbol{\omega}], \\ [d\boldsymbol{\omega},\boldsymbol{\omega}] &= -[\boldsymbol{\omega},d\boldsymbol{\omega}], \\ [[\boldsymbol{\omega},\boldsymbol{\omega}],\boldsymbol{\omega}] &= 0.\end{aligned} \tag{206}$$

Using Eq.(206) in Eq.(205) gives

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = [d\boldsymbol{\omega},\boldsymbol{\omega}]. \tag{207}$$

(ii) In Eq.(207) use Eq.(203) and the last equation in Eq.(206) to obtain

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = [\boldsymbol{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}]. \tag{208}$$

(iii) Use now the definition of the exterior covariant derivative [Eq.(192)] together with the fact that $\boldsymbol{\omega}(\mathbf{X^h}) = 0$, for all $\mathbf{X} \in T_pP$ to obtain

$$D^{\boldsymbol{\omega}}\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0.$$

We can then write the very important formula (known as the Bianchi identity),

$$D^{\boldsymbol{\omega}}\boldsymbol{\Omega}^{\boldsymbol{\omega}} = d\boldsymbol{\Omega}^{\boldsymbol{\omega}} + [\boldsymbol{\omega}, \boldsymbol{\Omega}^{\boldsymbol{\omega}}] = 0. \tag{209}$$

■

C.1 Local curvature in the Base Manifold M

Let (U, Φ) be a local trivialization of $\pi^{-1}(U)$ and s the associated cross section as defined above. Then, $s^*\boldsymbol{\Omega}^{\boldsymbol{\omega}} := \Omega^\omega$ (the pullback of $\boldsymbol{\Omega}^{\boldsymbol{\omega}}$) is a well defined 2-form field on U which takes values in the Lie algebra $\mathfrak{G}$. Let $\omega = s^*\boldsymbol{\omega}$ (see eq.(185)). If we recall that the differential operator d commutes with the pullback, we immediately get

$$\Omega^\omega = s^* D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\omega + \frac{1}{2}\left[\omega,\omega\right]. \tag{210}$$

It is convenient to define the *symbols*

$$\mathbf{D}\omega := s^* D^{\boldsymbol{\omega}}\boldsymbol{\omega}, \tag{211}$$

$$\mathbf{D}\Omega^\omega := s^* D^{\boldsymbol{\omega}}\boldsymbol{\Omega}^{\boldsymbol{\omega}} \tag{212}$$

and to write

$$\begin{aligned} \mathbf{D}\Omega^\omega &= 0, \\ \mathbf{D}\Omega^\omega &= d\Omega^\omega + [\omega,\Omega^\omega] = 0. \end{aligned} \tag{213}$$

Eq.(213) is also known as Bianchi identity.

Remark 40 *In gauge theories (Yang-Mills theories) Ω^ω is (except for numerical factors with physical units) called a* field strength *in the gauge Φ.*

Remark 41 *When G is a matrix group, as is the case in the presentation of gauge theories by physicists, Definition 35 of the commutator $[\boldsymbol{\varphi}, \boldsymbol{\psi}] \in \sec \overset{i+j}{\bigwedge}(P, \mathfrak{G})$ ($\boldsymbol{\varphi} \in \sec \overset{i}{\bigwedge}(P, \mathfrak{G})$, $\boldsymbol{\psi} \in \sec \overset{j}{\bigwedge}(P, \mathfrak{G})$) gives*

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = \boldsymbol{\varphi} \wedge \boldsymbol{\psi} - (-1)^{ij}\boldsymbol{\psi} \wedge \boldsymbol{\varphi}, \tag{214}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are considered as matrices of forms with values in $\mathfrak{G}$ and $\boldsymbol{\varphi} \wedge \boldsymbol{\psi}$ stands for the usual matrix multiplication. Then, when G is a matrix group, we can write Eqs.(203) and (210) as

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}, \tag{215}$$

$$\Omega^\omega := \mathbf{D}\omega = d\omega + \omega \wedge \omega. \tag{216}$$

C.2 Transformation of the Field Strengths Under a Change of Gauge

Consider two local trivializations (U, Φ) and (U', Φ') of P such that $p \in \pi^{-1}(U \cap U')$ has (x, g) and (x, g') as images in $(U \cap U') \times G$, where $x \in U \cap U'$. Let s, s' be the associated cross sections to Φ and Φ' respectively. By writing $s'^{*}\boldsymbol{\Omega^{\omega}} = \Omega^{\omega\prime}$, we have the following relation for the local curvature in the two different gauges such that $g' = hg$:

$$\Omega^{\omega\prime} = h\Omega^{\omega}h^{-1}, \quad \forall x \in U \cap U'. \tag{217}$$

We now give the *coordinate expressions* for the potential and field strengths in the trivialization Φ. Let $\langle x^{\mu} \rangle$ be a local chart for $U \subset M$ and let $\left\{\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}\right\}$ and $\{dx^{\mu}\}$, $\mu = 0, 1, 2, 3$, be (dual) bases of TU and $T^{*}U$ respectively. Then,

$$\omega = \omega^{a} \otimes \mathcal{G}_{a} = \omega^{a}_{\mu} dx^{\mu} \otimes \mathcal{G}_{a}, \tag{218}$$

$$\Omega^{\omega} = (\Omega^{\omega})^{a} \otimes \mathcal{G}_{a} = \frac{1}{2}\Omega^{a}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \otimes \mathcal{G}_{a}. \tag{219}$$

where ω^{a}_{μ}, $\Omega^{a}_{\mu\nu} : M \supset U \to \mathbb{R}$ (or $\mathbb{C}$) and we get

$$\Omega^{a}_{\mu\nu} = \partial_{\mu}\omega^{a}_{\nu} - \partial_{\nu}\omega^{a}_{\mu} + f^{a}_{bc}\omega^{b}_{\mu}\omega^{c}_{\nu}. \tag{220}$$

The following objects appear frequently in the presentation of gauge theories by physicists.

$$(\Omega^{\omega})^{a} = \frac{1}{2}\Omega^{a}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = d\omega^{a} + \frac{1}{2} f^{a}_{bc}\omega^{b} \wedge \omega^{c}, \tag{221}$$

$$\Omega^{\omega}_{\mu\nu} = \Omega^{a}_{\mu\nu}\mathcal{G}_{a} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} + [\omega_{\mu}, \omega_{\nu}], \tag{222}$$

$$\omega_{\mu} = \omega^{a}_{\mu}\mathcal{G}_{a}. \tag{223}$$

We now give the local expression of Bianchi identity. Using Eqs.(213) and (221) we have

$$\mathbf{D}\Omega^{\omega} := \frac{1}{2}(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu} = 0. \tag{224}$$

By putting

$$(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} := \mathbf{D}_{\rho}\Omega^{\omega}_{\mu\nu} \tag{225}$$

we have

$$\mathbf{D}_{\rho}\Omega^{\omega}_{\mu\nu} = \partial_{\rho}\Omega^{\omega}_{\mu\nu} + [\omega_{\rho}, \Omega^{\omega}_{\mu\nu}], \tag{226}$$

and

$$\mathbf{D}_{\rho}\Omega^{\omega}_{\mu\nu} + \mathbf{D}_{\mu}\Omega^{\omega}_{\nu\rho} + \mathbf{D}_{\nu}\Omega^{\omega}_{\rho\mu} = 0. \tag{227}$$

Physicists call the operator

$$\mathbf{D}_{\rho} := \partial_{\rho} + [\omega_{\rho}, \ ]. \tag{228}$$

the *covariant derivative*. The reason for this name will be given below.

C.3 Induced Connections

Let (P_1, M_1, π_1, G_1) and (P_2, M_2, π_2, G_2) be two principal bundles and let $\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism, i.e., f is fiber preserving, induces a diffeomorphism $f : M_1 \to M_2$ and there exists a homomorphism $\lambda : G_1 \to G_2$ such that for $g_1 \in G_1$, $p_1 \in P_1$ we have

$$\mathcal{F}(p_1 g_1) = R_{\lambda(g_1)} \mathcal{F}(p_1) \tag{229}$$

Proposition 42 *Let $\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism. Then a connection $\boldsymbol{\omega}_1$ on P_1 determines a unique connection on P_2.*

Remark 43 *Let $(P, M, \pi', \mathrm{O}(p,q)) = P_{\mathrm{O}(n)}(M)$ be the orthonormal frame bundle, which is as explained above a reduction of the frame bundle $F(M)$. Then, a connection on $P_{\mathrm{O}(n)}(M)$ determines a unique connection on $F(M)$. This is a very important result that is usually used implicitly.*

Proposition 44 *Let $F(M)$ be the frame bundle of a paracompact manifold M. Then, $F(M)$ can be reduced to a principal bundle with structure group $\mathrm{O}(n)$, and to each reduction there corresponds a Riemann metric field on M.*

Remark 45 *If M has dimension 4, and we substitute $\mathrm{O}(n) \mapsto \mathrm{SO}^e_{1,3}$ then to each reduction of $F(M)$ there corresponds a Lorentzian metric field on M.*

C.4 Linear Connections on a Manifold M

Definition 46 *A linear connection on a smooth manifold M is a connection $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n, \mathbb{R})$.*

Remark 47 *Given a Riemannian (Lorentzian) manifold $(M, \mathtt{g})$ a connection on $F(M)$ which is determined by a connection on the orthonormal frame bundle $P_{\mathrm{O}(n)}(M)$ ($P_{\mathrm{SO}^e_{1,3}}(M)$) is called a metric connection. After introducing the concept of covariant derivatives on vector bundles, we can show that the covariant derivative of the metric tensor with respect to a metric connection is null.*

Consider the mapping $\left. f \right|_p : T_x M \to \mathbb{R}^n$ (with $p = (x, \Sigma_x)$ in a given trivialization) which sends $\mathbf{v} \in T_x M$ into its components relative to the frame $\Sigma_x = \{ \left. e_1 \right|_x, ..., \left. e_n \right|_x \}$. Let $\{ \left. \theta^j \right|_x \}$ be the dual basis of $\{ \left. e_i \right|_x \}$. Then,

$$\left. f \right|_p (\mathbf{v}) = (\left. \theta^j \right|_x (\mathbf{v})) \tag{230}$$

Definition 48 *The canonical soldering form of M is the 1-form $\boldsymbol{\theta} \in \sec T^*F(M) \otimes \mathbb{R}^n$ such that for any $v \in \sec T_p F(M)$ such that $\mathbf{v} = \pi_* v$ we have*

$$\begin{aligned} (\boldsymbol{\theta}(v)) &:= \left. \boldsymbol{\theta}^a \right|_p (v) \mathbf{E}_a \\ &= \left. \theta^a \right|_x (\mathbf{v}) \mathbf{E}_a, \end{aligned} \tag{231}$$

where $\{\mathbf{E}_a\}$ is the canonical basis of $\mathbb{R}^n$ and $\{\boldsymbol{\theta}^a\}$ is a basis of $T^*F(M)$, with $\boldsymbol{\theta}^a = \pi^* \theta^a$, $\left. \boldsymbol{\theta}^a \right|_p (v) = \left. \theta^a \right|_x (\mathbf{v})$.

Definition 49 *The torsion form of a linear connection* $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n,\mathbb{R})$ *is the* 2*-form*

$$D\boldsymbol{\theta} = \boldsymbol{\Theta} \in \sec \bigwedge^2 T^*F(M) \otimes \mathbb{R}^n.$$

As it is easy to verify, the soldering form $\boldsymbol{\theta}$ and the torsion 2-form $\boldsymbol{\Theta}$ are tensorial of type $(\rho, \mathbb{R}^n)$, where $\rho(g) = g$, $g \in Gl(n,\mathbb{R})$.

We can show, using the same techniques used in the calculation of $D^{\boldsymbol{\omega}}\boldsymbol{\omega}(\mathbf{X_1},\mathbf{X_2})$ (Eq.(202)) that

$$\boldsymbol{\Theta} = d\boldsymbol{\theta} + [\boldsymbol{\omega},\boldsymbol{\theta}], \tag{232}$$

where [,] is the commutator product in the Lie algebra of the *affine* group $A(n,\mathbb{R}) = Gl(n,\mathbb{R}) \boxtimes \mathbb{R}^n$, where $\boxtimes$ means the *semi-direct* product. Suppose that $(\mathbf{E}_a^b, \mathbf{E}_c)$ is the canonical basis of $a(n,\mathbb{R})$, the Lie algebra of $A(n,\mathbb{R})$. Recalling that

$$\boldsymbol{\omega}(v) = \boldsymbol{\omega}_b^a(v)\mathbf{E}_a^b, \tag{233}$$

$$\boldsymbol{\theta}(v) = \boldsymbol{\theta}^a(v)\mathbf{E}_a, \tag{234}$$

we can show without difficulties that

$$D^{\boldsymbol{\omega}}\boldsymbol{\Theta} = [\boldsymbol{\Omega},\boldsymbol{\theta}] \tag{235}$$

C.5 Torsion and Curvature on M

Let $\{x^i\}$ be the coordinate functions associated to a local chart (U,φ) of the maximal atlas of M. Let $\Sigma \in \sec F(U)$ with $e_i = F_i^j \frac{\partial}{\partial x^j}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^a \mathbf{E}_a$. Take $\pi_* v = \mathbf{v}$. Then

$$\begin{aligned}(\boldsymbol{\theta}_p(v)) &= f|_p(\mathbf{v}) = f|_p(dx^j(\mathbf{v})\partial_j) = f|_p(dx^j(\mathbf{v})(F_j^k)^{-1}\mathbf{e}_k)\\ &= ((F_j^k)^{-1}dx^j(\pi_* v)).\end{aligned} \tag{236}$$

With this result it is quite obvious that given any $\mathbf{w} \in \mathbb{R}^n$, $\boldsymbol{\theta}$ determines a horizontal field $v_{\mathbf{w}} \in \sec TF(M)$ by

$$\boldsymbol{\theta}(v_{\mathbf{w}}(p)) = \mathbf{w}. \tag{237}$$

With these preliminaries we have the

Proposition 50 *There is a bijective correspondence between sections of* $T^*M \otimes T_s^r M$ *and sections of* $T^*F(M) \otimes \mathbb{R}^{n_q}$*, the space of tensorial forms of the type* $(\rho, \mathbb{R}^{n_q})$ *in* $F(M)$*, with* ρ *and* q *being determined by* $T_s^r M$ *.*

Using the above proposition and recalling that the soldering form is tensorial of type $(\rho(g), \mathbb{R}^n)$, $\rho(g) = g$, we see that it determines on M a vector valued differential 1-form $\theta = e_a \otimes dx^a \in \sec TM \otimes \bigwedge^1(T^*M)$. Also, the torsion $\boldsymbol{\Theta}$ is tensorial of type $(\rho(g), \mathbb{R}^n)$, $\rho(g) = g$ and thus define a vector valued 2-form on

M, $\Theta = e_a \otimes \Theta^a \in \sec TM \otimes \bigwedge^2(T^*M)$. We can show from Eq.(232) that given $u, w \in T_pF(M)$,

$$\Theta^a(\pi_*u, \pi_*w) = d\theta^a(\pi_*u, \pi_*w) + \omega^a_b(\pi_*u)\theta^b(\pi_*w) - \omega^a_b(\pi_*w)\theta^b(\pi_*u). \quad (238)$$

On the basis manifold this equation is often written:

$$\begin{aligned}\Theta = \mathbf{D}\theta &= e_a \otimes (\mathbf{D}\theta^a) \\ &= e_a \otimes (d\theta^a + \omega^a_b \wedge \theta^b),\end{aligned} \quad (239)$$

where we recognize $\mathbf{D}\theta^a$ as the exterior covariant derivative of index forms[21].

Also, the curvature $\mathbf{\Omega}^{\boldsymbol{\omega}}$ is tensorial of type $(\mathrm{Ad}, \mathbb{R}^{n^2})$. It then defines $\Omega = e_a \otimes \theta^b \otimes \Omega^a_b \in \sec T^1_1 M \otimes \bigwedge^2(T^*M)$ which is easily find to be

$$\begin{aligned}\Omega &= e_a \otimes \theta^b \otimes \Omega^a_b \\ &= e_a \otimes \theta^b \otimes (d\omega^a_b + \omega^a_c \wedge \omega^c_b),\end{aligned} \quad (240)$$

where the $\Omega^a_b \in \sec \bigwedge^2(T^*M)$ are the curvature 2-forms.

$$\Omega^a_b := d\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (241)$$

Note that sometimes the *symbol* $\mathbf{D}\omega^a_b$ such that $\Omega^a_b := \mathbf{D}\omega^a_b$ is introduced in some texts. Of course, the symbol $\mathbf{D}$ cannot be interpreted in this case as the exterior covariant derivative of index forms. This is expected since $\boldsymbol{\omega} \in \sec \bigwedge^1(T^*P(M)) \otimes gl(n, \mathbb{R})$ is *not* tensorial.

D Covariant Derivatives on Vector Bundles

Consider a vector bundle $(E, M, \pi_1, G, \mathbf{V})$ denoted $E = P \times_\rho \mathbf{V}$ *associated* to a *PFB* bundle (P, M, π, G) by the linear representation ρ of G in the vector space $\mathbf{V}$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Also, let $\dim_{\mathbb{F}} \mathbf{V} = m$. Consider again the trivializations of P and E given by Eqs.(159)-(161). Then, we have the

Definition 51 *The* parallel transport *of $\mathbf{\Psi_0} \in E$, $\pi_1(\mathbf{\Psi_0}) = x_0$, along the curve $\sigma : \mathbb{R} \ni I \to M$, $t \mapsto \sigma(t)$ from $x_0 = \sigma(0) \in M$ to $x = \sigma(t)$ is the element $\mathbf{\Psi}_{\parallel t} \in E$ such that:*

(i) $\pi_1(\mathbf{\Psi}_{\parallel t}) = x$,
(ii) $\chi_i(\mathbf{\Psi}_{\parallel t}) = \rho(\varphi_i(p_{\parallel t}) \circ \varphi_i^{-1}(p_0))\chi_i(\mathbf{\Psi_0})$.
(iii) $p_{\parallel t} \in P$ is the parallel transport of $p_0 \in P$ along σ from x_0 to x as defined in Eq.(176) above.

[21] Rigorously speaking, if Eq.(239) is to agree with Eq.(238) we must have $\bar{\omega}^a_b \wedge \bar{\theta}^b = (\bar{\omega}^a_b \otimes \bar{\theta}^b - \bar{\theta}^b \otimes \bar{\omega}^a_b)$. This is not the definition of the exterior product we used in Section 2 because a factor 1/2 is missing. However, this causes no troubles in the calculations we did using the Clifford bundle formalism.

Definition 52 *Let Y be a vector at x_0 tangent to the curve σ (as defined above). The covariant derivative of $\mathbf{\Psi} \in \sec E$ in the direction of Y is denoted $(D_Y^E \mathbf{\Psi})_{x_0} \in \sec E$ and*

$$(D_Y^E \mathbf{\Psi})(x_0) \equiv (D_Y^E \mathbf{\Psi})_{x_0} = \lim_{t \to 0} \frac{1}{t} (\mathbf{\Psi}^0_{\| t} - \mathbf{\Psi_0}), \tag{242}$$

where $\mathbf{\Psi}^0_{\| t}$ is the "vector" $\mathbf{\Psi}_t \equiv \mathbf{\Psi}(\sigma(t))$ of a section $\mathbf{\Psi} \in \sec E$ parallel transported along σ from $\sigma(t)$ to x_0, the only requirement on σ being

$$\left. \frac{d}{dt} \sigma(t) \right|_{t=0} = Y. \tag{243}$$

In the local trivialization (U_i, Ξ_i) of E (see Eqs.(159)-(161)) if Ψ_t is the element in $\mathbf{V}$ representing $\mathbf{\Psi}_t$, we have

$$\chi_i(\Psi^0_{\| t}) = \rho(g_0 g_t^{-1}) \chi_{i|\sigma(t)}(\Psi_t). \tag{244}$$

By choosing p_0 such that $g_0 = e$ we can compute Eq.(242):

$$\begin{aligned} (D_Y^E \mathbf{\Psi})_{x_0} &= \left. \frac{d}{dt} \rho(g^{-1}(t) \Psi_t) \right|_{t=0} \\ &= \left. \frac{d\Psi_t}{dt} \right|_{t=0} - \left(\rho'(e) \left. \frac{dg(t)}{dt} \right|_{t=0} \right) (\Psi_0). \end{aligned} \tag{245}$$

This formula is trivially generalized for the covariant derivative in the direction of an arbitrary vector field $Y \in \sec TM$.

With the aid of Eq.(245) we can calculate, e.g., the covariant derivative of $\mathbf{\Psi} \in \sec E$ in the direction of the vector field $Y = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu$. This covariant derivative is denoted $D^E_{\partial_\mu} \mathbf{\Psi}$.

We need now to calculate $\left. \frac{dg(t)}{dt} \right|_{t=0}$. In order to do that, recall that if $\frac{d}{dt}$ is a tangent to the curve σ in M, then $s_* \left(\frac{d}{dt} \right)$ is a tangent to $\hat{\sigma}$ the horizontal lift of σ, i.e., $s_* \left(\frac{d}{dt} \right) \in HP \subset TP$. As defined before $s = \Phi_i^{-1}(x, e)$ is the cross section associated to the trivialization Φ_i of P (see Eq.(158)). Then, as g is a mapping $U \to G$ we can write

$$\left[s_* (\frac{d}{dt}) \right] (g) = \frac{d}{dt} (g \circ \sigma). \tag{246}$$

To simplify the notation, introduce local coordinates $\langle x^\mu, g \rangle$ in $\pi^{-1}(U)$ and write $\sigma(t) = (x^\mu(t))$ and $\hat{\sigma}(t) = (x^\mu(t), g(t))$. Then,

$$s_* \left(\frac{d}{dt} \right) = \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu} + \dot{g}(t) \frac{\partial}{\partial g}, \tag{247}$$

in the local coordinate basis of $T(\pi^{-1}(U))$. An expression like the second member of Eq.(247) defines in general a vector tangent to P but, according to its definition, $s_* \left(\frac{d}{dt} \right)$ is in fact horizontal. We must then impose that

$$s_* \left(\frac{d}{dt} \right) = \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu} + \dot{g}(t) \frac{\partial}{\partial g} = \alpha^\mu \left(\frac{\partial}{\partial x^\mu} + \omega^a_\mu \mathcal{G}_a g \frac{\partial}{\partial g} \right), \tag{248}$$

for some α^{μ}.

We used the fact that $\frac{\partial}{\partial x^{\mu}}+\omega_{\mu}^{a}\mathcal{G}_{a}g\frac{\partial}{\partial g}$ is a basis for HP, as can easily be verified from the condition that $\boldsymbol{\omega}(Y^{h})=0$, for all $Y\in HP$. We immediately get that

$$\alpha^{\mu}=\dot{x}^{\mu}(t), \tag{249}$$

and

$$\frac{dg(t)}{dt}=\dot{g}(t)=-\dot{x}^{\mu}(t)\omega_{\mu}^{a}\mathcal{G}_{a}g, \tag{250}$$

$$\left.\frac{dg(t)}{dt}\right|_{t=0}=-\dot{x}^{\mu}(0)\omega_{\mu}^{a}\mathcal{G}_{a}. \tag{251}$$

With this result we can rewrite Eq.(245) as

$$(D_{Y}^{E}\boldsymbol{\Psi})_{x_{0}}=\left.\frac{d\Psi_{t}}{dt}\right|_{t=0}-\rho'(e)\omega(Y)(\Psi_{0}),\qquad Y=\left.\frac{d\sigma}{dt}\right|_{t=0}, \tag{252}$$

which generalizes trivially for the covariant derivative along a vector field $Y\in\sec TM$.

Remark 53 *Many texts introduce the covariant derivative operator D_{Y}^{E} acting on sections of the vector bundle E as follows:*

Definition 54 *A connection D^{E} on M is a mapping*

$$\begin{aligned} D^{E} &: \sec TM\times\sec E\rightarrow\sec E,\\ (X,\boldsymbol{\Psi}) &\mapsto D_{Y}^{E}\boldsymbol{\Psi}. \end{aligned} \tag{253}$$

such that $D_{Y}^{E}:\sec E\rightarrow\sec E$ satisfies the following properties:

$$\begin{array}{cc} (i) & D_{X}^{E}(a\boldsymbol{\Psi})=a(D_{X}^{E}\boldsymbol{\Psi}),\\ (ii) & D_{X}^{E}(\boldsymbol{\Psi}+\boldsymbol{\Phi})=D_{X}^{E}\boldsymbol{\Psi}+D_{X}^{E}\boldsymbol{\Phi},\\ (iii) & D_{X}^{E}(f\boldsymbol{\Psi})=X(f)+fD_{X}^{E}\boldsymbol{\Psi},\\ (iv) & D_{X+Y}^{E}\boldsymbol{\Psi}=D_{X}^{E}\boldsymbol{\Psi}+D_{Y}^{E}\boldsymbol{\Psi},\\ (v) & D_{fX}^{E}\boldsymbol{\Psi}=fD_{X}^{E}\boldsymbol{\Psi}. \end{array} \tag{254}$$

$\forall$ $X,Y\in\sec TM$, $\boldsymbol{\Psi},\boldsymbol{\Phi}\in\sec E$, $\forall a\in\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ (the field of scalars entering the definition of the vector space $\mathbf{V}$ of E) $\forall f\in C^{\infty}(M)$, where $C^{\infty}(M)$ is the set of smooth functions with values in $\mathbb{F}$.

Of course, all properties in Eq.(254) follows directly from Eq.(252). However, the point of view encoded in Definition 54 may be appealing to physicists. To see, first recall that $E=P\times_{\rho}\mathbf{V}$. Recall that ϱ stands for the representation of G in the vector space $\mathbf{V}$.

Definition 55 *The dual bundle of E is the bundle is the bundle $E^{*}=P\times_{\rho^{*}}\mathbf{V}^{*}$, where $\mathbf{V}^{*}$ is the dual space of $\mathbf{V}$ and ϱ^{*} is the representation of G in the vector space $\mathbf{V}^{*}$.*

Example 56 *As examples, recall that the tangent bundle is* $TM = F(M) \times_{\rho} \mathbb{R}^n$ *where* $\rho : Gl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$ *denotes the standard representation. Then,* $T^*M = F(M) \times_{\rho^*} (\mathbb{R}^n)^*$ *and the dual representation* ρ^* *satisfies* $\rho^*(g) = \rho(g^{-1})^t$. *Also, the tensor bundle of tensors of type* (r,s), *the bundle of homogenous* k-*vectors and the bundle of homogeneous* k-*forms are:*

$$\begin{aligned} T_s^r M &= \bigotimes\nolimits_s^r TM = F(M) \times_{\otimes_s^r} (\bigotimes\nolimits_s^r \mathbb{R}^n), \\ \bigwedge\nolimits^k (TM) &= F(M) \times_{\bigwedge_\rho^k} \bigwedge\nolimits^k \mathbb{R}^n, \\ \bigwedge\nolimits^k (T^*M) &= F(M) \times_{\bigwedge_{\rho^*}^k} \bigwedge\nolimits^k \mathbb{R}^n, \end{aligned} \tag{255}$$

where $\bigotimes_s^r$, $\bigwedge_\rho^k$ *and* $\bigwedge_{\rho^*}^k$ *are the induced tensor product and exterior powers representations.*

Definition 57 *The bundle* $E \otimes E^*$ *is called the bundle of endomorphisms of* E *and will be denoted by* $\mathrm{End}(E)$.

Definition 58 *A connection* D^{E^*} *acting on* E^* *is defined by*

$$(D_X^{E^*} \boldsymbol{\Upsilon}^*)(\boldsymbol{\Psi}) = X\left(\boldsymbol{\Upsilon}^*(\boldsymbol{\Psi})\right) - \boldsymbol{\Upsilon}^*(D_X^E \boldsymbol{\Psi}), \tag{256}$$

$\forall \boldsymbol{\Upsilon}^* \in \sec E^*$, $\forall \boldsymbol{\Psi} \in \sec E$ *and* $\forall X \in \sec TM$.

Definition 59 *A connection* $D^{E \otimes E^*}$ *acting on sections of* $E \otimes E^*$ *is defined* $\forall \boldsymbol{\Upsilon}^* \in \sec E^*$, $\forall \boldsymbol{\Psi} \in \sec E$ *and* $\forall X \in \sec TM$ *by*

$$D_X^{E \otimes E^*} \boldsymbol{\Upsilon}^* \otimes \boldsymbol{\Psi} = D_X^{E^*} \boldsymbol{\Upsilon}^* \otimes \boldsymbol{\Psi} + \boldsymbol{\Upsilon}^* \otimes D_X^E \boldsymbol{\Psi}. \tag{257}$$

We shall abbreviate $D^{E \otimes E^*}$ by $D^{\mathrm{End}E}$. Eq.(257) may be generalized in an obvious way in order to define a connection on arbitrary tensor products of bundles $E \otimes E' \otimes ... \otimes E'^{\cdots\prime}$. Finally, we recall for completeness that given two bundles, say E and E' and given connections D^E and $D^{E'}$ there is an obvious connection $D^{E \oplus E'}$ defined in the Whitney bundle $E \oplus E'$ (recall definition 19) It is given by

$$D_X^{E \oplus E'} (\boldsymbol{\Psi} \oplus \boldsymbol{\Psi}') = D_X^E \boldsymbol{\Psi} \oplus D_X^{E'} \boldsymbol{\Psi}', \tag{258}$$

$\forall \boldsymbol{\Psi} \in \sec E$, $\forall \boldsymbol{\Psi}' \in \sec E'$ and $\forall X \in \sec TM$.

D.1 Connections on E over a Lorentzian Manifold

In what follows we suppose that $(M, \mathbf{g})$ is a Lorentzian manifold. We recall that the manifold M in a Lorentzian structure is supposed paracompact. Then, according to Proposition 12 the bundles E, E^*, $T_s^r M$ and $\mathrm{End}E$ admit global cross sections.

We then write for the covariant derivative of $\boldsymbol{\Psi} \in \sec E$ and $X \in \sec TM$,

$$D_X^E \boldsymbol{\Psi} = D_X^{0E} \boldsymbol{\Psi} + \mathcal{W}(X) \boldsymbol{\Psi}, \tag{259}$$

where $\mathcal{W} \in \sec \mathrm{End}E \otimes T^*M$ will be called *connection* 1-form (or *potential)* for D_X^E and D_X^{0E} is a well-defined connection on E, that we are going to determine.

Consider then a open set $U \subset M$ and a trivialization of E in U. Such a trivialization is said to be a *choice of a gauge.*

Let $\{\mathbf{E}_i\}$ be the canonical basis of $\mathbf{V}$. Let $\mathbf{\Psi}|_U \in \sec\ E|_U = \pi^{-1}(U)$. Consider the trivialization $\Xi : \pi^{-1}(U) \to U \times \mathbf{V}$, $\Xi(\mathbf{\Psi}) = (\pi(\mathbf{\Psi}), \chi(\mathbf{\Psi})) = (x, \chi(\mathbf{\Psi}))$. In this trivialization we write

$$\mathbf{\Psi}|_U := (x,\Psi(x)), \tag{260}$$

$\Psi(x) \in \mathbf{V}$, $\forall x \in U$, with $\Psi : U \to \mathbf{V}$ a smooth function. Let $\{s_i\} \in \sec\ E|_U$, $s_i = \chi^{-1}(\mathbf{E}_i)$, $i = 1, 2, ..., m$ be a basis of sections of $E|_U$ and $\{e_\mu\} \in \sec F(U), \mu = 0, 1, 2, 3$ a basis for TU. Let also $\{\varepsilon^\nu\}$, $\varepsilon^\nu \in \sec T^*U$, be the dual basis of $\{e_\mu\}$ and $\{s^{*i}\} \in \sec\ E^*|_U$, be a basis of sections of $E^*|_U$ dual to the basis $\{s_i\}$.

We define the connection coefficients in the chosen gauge by

$$D_{e_\mu}^E s_i = \mathcal{W}_{\mu i}^j s_j. \tag{261}$$

Then, if $\Psi = \Psi^i s_i$ and $X = X^\mu e_\mu$

$$\begin{aligned} D_X^E \Psi &= X^\mu D_{e_\mu}^E (\Psi^i s_i) \\ &= X^\mu \left[e_\mu(\Psi^i) + \mathcal{W}_{\mu j}^i \Psi^j \right] s_i. \end{aligned} \tag{262}$$

Now, let us concentrate on the term $X^\mu \mathcal{W}_{\mu j}^i \Psi^j s_i$. It is, of course a new section $F := (x, X^\mu \mathcal{W}_{\mu j}^i \Psi^j s_i)$ of $E|_U$ and $X^\mu \mathcal{W}_{\mu j}^i \Psi^j s_i$ is linear in both X and Ψ.

This observation shows that $\mathcal{W}^U \in \sec(\mathrm{End}\ E|_U) \otimes T^*U$, such that in the trivialization introduced above is given by

$$\mathcal{W}^U = \mathcal{W}_{\mu j}^i s_j \otimes s^{*i} \otimes \varepsilon^\mu \tag{263}$$

is the representative of $\mathcal{W}$ in the chosen gauge.

Note that if $X \in \sec TU$ and $\mathbf{\Psi} := (x,\Psi(x)) \in \sec\ E|_U$ we have

$$\begin{aligned} \omega^U(X) &:= \omega_X^U = X^\mu \mathcal{W}_{\mu j}^i s_j \otimes s^{*i}, \\ \omega_X^U(\Psi) &= X^\mu \mathcal{W}_{\mu j}^i \Psi^i s_j. \end{aligned} \tag{264}$$

We can then write

$$D_X^E \Psi = X(\Psi) + \omega_X^U(\Psi), \tag{265}$$

thereby identifying $D_X^{0E} \Psi = X(\Psi)$. In this case D_X^{0E} is called the standard *flat* connection.

Now, we can state a very important result which has been used in Chapter 2 to write the different decompositions of Riemann-Cartan connections.

Proposition 60 *Let* $\mathbf{D}^{0E}$ *and* $\mathbf{D}^E$ *be arbitrary connections on* E*. Then there exists* $\bar{\mathcal{W}} \in \sec \mathrm{End}E \otimes T^*M$ *such that for any* $\mathbf{\Psi} \in \sec E$ *and* $X \in \sec TM$,

$$\mathbf{D}_X^E \mathbf{\Psi} = \mathbf{D}_X^{0E} \mathbf{\Psi} + \bar{\mathcal{W}}(X) \mathbf{\Psi}. \tag{266}$$

D.2 Gauge Covariant Connections

Definition 61 *A connection D^E on E is said to be a G-connection if for any $g \in G$ and any $\mathbf{\Psi} \in \sec E$ there exists a connection $\mathbf{D}'^E$ on E such that for any $X \in \sec TM$*

$$D'^E_X\,(\rho(g)\mathbf{\Psi}) = \rho(g) D^E_X \mathbf{\Psi}. \tag{d11}$$

Proposition 62 *If $D^E_X \mathbf{\Psi} = D^{0E}_X \mathbf{\Psi} + \bar{\mathcal{W}}(X)\mathbf{\Psi}$ for $\mathbf{\Psi} \in \sec E$ and $X \in \sec TM$, then $D'^E_X \mathbf{\Psi} = D^{0E}_X \mathbf{\Psi} + \bar{\mathcal{W}}'(X)\mathbf{\Psi}$ with*

$$\bar{\mathcal{W}}'(X) = g\bar{\mathcal{W}}(X)g^{-1} + gdg^{-1}. \tag{267}$$

Suppose that the vector bundle E has the same structural group as the orthonormal frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$, which as we know is a reduction of the frame bundle $F(M)$. In this case we give the

Definition 63 *A connection $\mathbf{D}^E$ on E is said to be a generalized G-connection if for any $g \in G$ and any $\mathbf{\Psi} \in \sec E$ there exists a connection $\mathbf{D}'^E$ on E such that for any $X \in \sec TM$, $TM = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\rho^{TM}} \mathbb{R}^4$*

$$D'^E_{X'}\,(\rho(g)\mathbf{\Psi}) = \rho(g) D^E_X \mathbf{\Psi}, \tag{268}$$

where $X' = \rho^{TM} X \in \sec TM$.

D.3 Curvature Again

Definition 64 *Let D^E be a G-connection on E. The curvature operator $\mathbf{R}^E \in \sec \bigwedge^2 T^*M \otimes$ End$E$ of D^E is the mapping*

$$\mathbf{R}^E : \sec TM \otimes TM \otimes E \to E, \tag{269}$$

$$\mathbf{R}^E(X,Y)\mathbf{\Psi} = D^E_X D^E_Y \mathbf{\Psi} - D^E_Y D^E_X \mathbf{\Psi} - D^E_{[X,Y]}\mathbf{\Psi}$$

$$\mathbf{R}^E(X,Y) = D^E_X D^E_Y - D^E_Y D^E_X - D^E_{[X,Y]}, \tag{d14}$$

for any $\mathbf{\Psi} \in \sec E$ and $X, Y \in \sec TM$.

If $X = \partial_\mu, Y = \partial_\nu \in \sec TU$ are coordinate basis vectors associated to the coordinate functions $\{x^\mu\}$ we have

$$\mathbf{R}^E(\partial_\mu, \partial_\nu) := \mathbf{R}^E_{\mu\nu} = \left[D^E_{\partial_\mu}, D^E_{\partial_\nu}\right] \tag{270}$$

In a local basis $\{s_i \otimes s^{*j}\}$ of End$E$ we have under the local trivialization used above

$$\begin{aligned} \mathbf{R}^E_{\mu\nu} &= \mathbf{R}^a_{\mu\nu b} s_a \otimes s^{*b}, \\ \mathbf{R}^a_{\mu\nu b} &= \partial_\mu \mathcal{W}^a_{\nu b} - \partial_\nu \mathcal{W}^a_{\mu b} + \mathcal{W}^a_{\mu c}\mathcal{W}^c_{\nu b} - \mathcal{W}^a_{\nu c}\mathcal{W}^c_{\mu b} \end{aligned} \tag{271}$$

Eq.(271) can also be written

$$\mathbf{R}^E_{\mu\nu} = \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + [\mathcal{W}_\mu, \mathcal{W}_\nu] \tag{272}$$

D.4 Exterior Covariant Derivative Again

Definition 65 *Consider* $\Psi\otimes A_r \in \sec E\otimes\bigwedge^r T^*M$ *and* $B_s \in \bigwedge^s T^*M$ *. We define* $(\Psi\otimes A_r)\otimes_\wedge B_s$ *by*

$$(\Psi\otimes A_r)\otimes_\wedge B_s = \Psi\otimes(A_r\wedge B_s) \tag{273}$$

Definition 66 *Let* $\Psi\otimes A_r \in \sec E\otimes\bigwedge^r T^*M$ *and* $\mathbf{\Pi}\otimes B_s \in \sec \mathrm{End}E\otimes\bigwedge^s T^*M$. *We define* $(\Pi\otimes B_s)\otimes_\wedge(\Psi\otimes A_r)$ *by*

$$(\Pi\otimes B_s)\otimes_\wedge(\Psi\otimes A_r) = \Pi(\Psi)\otimes(B_s\wedge A_r) \tag{274}$$

Definition 67 *Given a connection* D^E *acting on* E, *the exterior covariant derivative* $\boldsymbol{d}^{D^E}$ *acting on sections of* $E\otimes\bigwedge^r T^*M$ *and the exterior covariant derivative* $\boldsymbol{d}^{D^{\mathrm{EndE}}}$ *sections of* $\mathrm{End}E\otimes\bigwedge^s T^*M$ $(r,s=0,1,2,3,4)$ *are given by*

(i) if $\mathbf{\Psi}\in\sec E$ *then for any* $X,Y\in\sec TM$

$$\mathbf{d}^{D^E}\Psi(X) = D^E\Psi, \tag{275}$$

(ii) For any $\Psi\otimes A_r \in \sec E\otimes\bigwedge^r T^*M$

$$\mathbf{d}^{D^E}(\Psi\otimes A_r) = \mathbf{d}^{D^E}\Psi\otimes\wedge A_r + \Psi\otimes dA_r, \tag{276}$$

(iii) For any $\mathbf{\Pi}\otimes B_s \in \sec\mathrm{End}E\otimes\bigwedge^s T^*M$

$$\mathbf{d}^{D^{\mathrm{EndE}}}(\Pi\otimes B_s) = \mathbf{d}^{D^{\mathrm{EndE}}}\Pi\otimes_\wedge B_s + \Pi\otimes dB_s, \tag{277}$$

Proposition 68 *Consider the bundle product* $\mathfrak{E} = (\mathrm{End}E\otimes\bigwedge^s T^*M)\otimes_\wedge E\otimes\bigwedge^r(T^*M)$. *Let* $\mathbf{\Pi} = \Pi\otimes B_s \in \sec\mathrm{End}E\otimes\bigwedge^s T^*M$ *and* $\mathbf{\Psi} = \Psi\otimes A_r \in \sec E\otimes\bigwedge^r T^*M$. *Then the exterior covariant derivative* $\mathbf{d}^{D^{\mathfrak{E}}}$ *acting on sections of* $\mathfrak{E}$ *satisfies*

$$\mathbf{d}^{D^{\mathfrak{E}}}(\mathbf{\Pi}\otimes_\wedge\mathbf{\Psi}) = (\mathbf{d}^{D^{\mathrm{EndE}}}\mathbf{\Pi})\otimes_\wedge\mathbf{\Psi} + (-1)^s\mathbf{\Pi}\otimes_\wedge\mathbf{d}^{D^E}\mathbf{\Psi}. \tag{278}$$

Exercise 69 *We can now show several interesting results, which make contact with results obtained earlier when we analyzed the connections and curvatures on principal bundles and which allowed us sometimes the use of sloppy notations in the main text:*

(i) Suppose that the bundle admits a flat connection D^{0E}. *We put* $\mathbf{d}^{D^{0E}} = d$. *Then, if* $\chi\in\sec E\otimes\bigwedge^r T^*M$ *we have*

$$\mathbf{d}^{D^{0E}}\chi = \mathbf{d}\chi + \mathcal{W}\otimes_\wedge\chi$$

(ii) If $\chi\in\sec E\otimes\bigwedge^r T^*M$ *we have*

$$(\mathbf{d}^{D^E})^2\chi = \mathbf{R}^E\otimes_\wedge\chi. \tag{279}$$

(iii) If $\chi\in\sec E\otimes\bigwedge^r T^*M$ *we have*

$$(\mathbf{d}^{D^E})^3\chi = \mathbf{R}^E\otimes_\wedge\mathbf{d}^{D^E}\chi \tag{280}$$

(iii) Suppose that the bundle admits a flat connection D^{0E}. We put $\mathbf{d}^{D^{0E}} = d$. Then, if

(iv) If $\mathbf{\Pi} \in \sec \mathrm{End} E \otimes \bigwedge^s T^*M$

$$\mathbf{d}^{D^{\mathrm{End}E}}\mathbf{\Pi} = \mathbf{d\Pi} + [\mathcal{W}, \mathbf{\Pi}] \tag{281}$$

(v)

$$\mathbf{d}^{D^{\mathrm{End}E}}\mathbf{R}^E = 0 \tag{282}$$

(vi)

$$\mathbf{R}^E = d\mathcal{W} + \mathcal{W} \otimes_\wedge \mathcal{W} \tag{283}$$

Remark 70 *Note that $\mathbf{R}^E \neq \mathbf{d}^{D^{\mathrm{End}E}}\mathcal{W}$.*

We hope that the material presented in the Appendix be enough to permit our reader to follow the more difficult parts of the text and in particular to see the reason for or use of many eventual sloppy notations.

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